

TWISTED VERTEX REPRESENTATIONS AND SPIN CHARACTERS

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ABSTRACT. We establish a new group-theoretic realization of the basic representations of the twisted affine and twisted toroidal algebras of ADE types in the same spirit of our new approach to the McKay correspondence. Our vertex operator construction provides a unified description to the character tables for the spin cover of the wreath product of the twisted hyperoctahedral groups and an arbitrary finite group.

INTRODUCTION

The connection between the homogeneous vertex representations of affine Lie algebras and the wreath products $\Gamma_n = \Gamma^n \rtimes S_n$ associated to finite subgroups Γ of $SL_2(\mathbb{C})$ was first pointed out in [W1] and subsequently established fully in [FJW1]. This initiates a new approach to the McKay correspondence (cf. [Mc]), which classically relates the finite subgroups of $SL_2(\mathbb{C})$ in a bijective manner with affine Dynkin diagrams of ADE type.

These results have been further extended in [FJW2] to realize the vertex representations of twisted affine Lie algebras $\widehat{\mathfrak{g}}[-1]$ and its toroidal counterpart by using the spin representations of a double cover $\widetilde{\Gamma}_n = \Gamma^n \rtimes \widetilde{S}_n$ of the wreath product Γ_n . Here \widetilde{S}_n is a double cover of the symmetric group S_n whose representation theory was developed by Schur [S] and reformulated in terms of twisted vertex operators in [J1]. The *algebraic* construction of the vertex representation of the twisted affine algebra $\widehat{\mathfrak{g}}[-1]$ was obtained first in [LW] in the case of $\mathfrak{g} = sl_2$, and in [FLM1, FLM2] for general \mathfrak{g} as one ingredient in the vertex (operator) algebra construction of the Monster group.

The goal of this paper is to provide a new finite-group-theoretic realization of the vertex representations of the twisted affine Lie algebra $\widehat{\mathfrak{g}}[-1]$ and its toroidal counterpart. Instead of $\widetilde{\Gamma}_n$ we will use a semi-direct product $\widetilde{H}\Gamma_n = \Gamma^n \rtimes \widetilde{H}_n$, where \widetilde{H}_n is a double cover of the

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hyperoctahedral group H_n . The finite group $\tilde{H}\Gamma_n$ can also be thought as a double cover of the wreath product $(\Gamma \times \mathbb{Z}_2)^n \rtimes S_n$.

Our present construction recaptures all the constructions in [FJW2] with additional advantages. First, this has a natural generalization given in a companion paper [W2], which is analogous to [W1], to the K -theory setup and it is intimately related to geometry. Secondly, the present constructions of the twisted vertex operators, Heisenberg algebra, and the characteristic map etc are essentially over \mathbb{Z} while those in [FJW2] involve an inevitable square root of 2 which originates in the theory of spin representations of \tilde{S}_n . Our present construction in principle paves the way to study the quantum twisted vertex representations at roots of unity which we will investigate elsewhere.

More explicitly, we take the supermodule approach of Jozéfiak [Jo1] to study the spin representations of $\tilde{H}\Gamma_n$ for an arbitrary finite group Γ . We give a description with a complete proof of the *split* conjugacy classes in $\tilde{H}\Gamma_n$ which play an important role in understanding the spin supermodules of $\tilde{H}\Gamma_n$. This generalizes earlier works on \tilde{H}_n and its spin representations (cf. [Sg, Jo2, St]). This result was also stated in Read [R] when Γ is cyclic.

Given a finite group Γ , we consider a direct sum over all n , denoted by $R_{H\Gamma}^-$, of the Grothendieck groups of *spin* supermodules of $\tilde{H}\Gamma_n$. We show that $R_{H\Gamma}^-$ carries a natural Hopf algebra structure. Associated to a self-dual virtual character ξ of Γ we define a ξ -weighted symmetric bilinear form on $R_{H\Gamma}^-$ (compare [FJW1, FJW2]). The space $R_{H\Gamma}^-$ can be shown to be isomorphic to a Fock space of a twisted Heisenberg algebra. The twisted vertex operators also make a natural appearance. One can further interpret the Fock space as a distinguished space of symmetric functions parametrized by the set of irreducible characters of Γ , and in this way the irreducible characters of $\tilde{H}\Gamma_n$ correspond essentially to the Schur Q -functions.

When Γ is a subgroup of $SL_2(\mathbb{C})$ and the weight ξ is chosen suitably, the Fock space $R_{H\Gamma}^-$ leads to a construction of the twisted vertex representations of $\hat{\mathfrak{g}}[-1]$ and its toroidal counterpart. On the other hand, when choosing the weight ξ to be trivial, the twisted vertex operator approach allows us to compute spin characters of all irreducible supermodules of the group $\tilde{H}\Gamma_n$, as done for \tilde{S}_n in [J1] and for $\tilde{\Gamma}_n$ in [FJW2].

The layout of this paper is as follows. In Sect. 1 we studied in detail the conjugacy classes of the group $\tilde{H}\Gamma_n$. In Sect. 2, we introduce the Grothendieck group $R^-(\tilde{H}\Gamma_n)$ and a weighted bilinear form on it. We construct the Hopf algebra structure on $R_{H\Gamma}^-$. In Sect. 3 we identify

$R_{H\Gamma}^-$ with a Fock space of a twisted Heisenberg algebra. In Sect. 4 we construct twisted vertex operators (essentially on $R_{H\Gamma}^-$) in terms of group theoretic operators. When Γ is a finite subgroup of $SL_2(\mathbb{C})$, this leads to a group theoretic construction of the vertex representation of twisted affine and toroidal Lie algebras. In Sect. 5 we construct the irreducible spin super characters of $\tilde{H}\Gamma_n$ and recover their character table from vertex operator viewpoint. When the statements can be proved similarly as in [FJW2], we often sketch only or omit the proofs and refer the reader to *loc. cit.* for more detail.

1. THE CONJUGACY CLASSES OF THE FINITE GROUPS $\tilde{H}\Gamma_n$

In this section we introduce the finite group $\tilde{H}\Gamma_n$ and a natural \mathbb{Z}_2 -grading on it. We also classify the so-called *split* conjugacy classes in $\tilde{H}\Gamma_n$ which will play a key role in the study of spin supermodules of $\tilde{H}\Gamma_n$ in later sections.

1.1. Definitions. Let Π_n be the finite group generated by a_i ($i = 1, \dots, n$) and the central element z subject to the relations

$$(1.1) \quad a_i^2 = z, \quad z^2 = 1, \quad a_i a_j = z a_j a_i \quad (i \neq j).$$

The symmetric group S_n acts on Π_n by $s(a_i) = a_{s(i)}$, $s \in S_n$. The semidirect product $\tilde{H}_n := \Pi_n \rtimes S_n$ is called the *twisted hyperoctahedral group*. Explicitly the multiplication in \tilde{H}_n is given by

$$(a, s)(a', s') = (as(a'), ss'), \quad a, a' \in \Pi_n, s, s' \in S_n.$$

Since $\Pi_n/\{1, z\} \simeq \mathbb{Z}_2^n$, the group \tilde{H}_n is a double cover of the hyperoctahedral group $H_n := \mathbb{Z}_2^n \rtimes S_n$.

Let Γ be a finite group. The twisted hyperoctahedral group \tilde{H}_n acts on the product group $\Gamma^n := \Gamma \times \dots \times \Gamma$ by letting Π_n act trivially on Γ^n and letting S_n act by $s(g_1, \dots, g_n) = (g_{s^{-1}(1)}, \dots, g_{s^{-1}(n)})$, $s \in S_n$. The finite group $\tilde{H}\Gamma_n$ is then defined to be the semi-direct product of \tilde{H}_n and Γ^n . Alternatively, the symmetric group S_n naturally acts on $\Gamma^n \times \Pi_n$ by simultaneous permutations of elements in Γ^n and Π_n , and we may regard $\tilde{H}\Gamma_n$ as the semi-direct product of the symmetric group S_n and $\Gamma^n \times \Pi_n$.

The double covering \tilde{H}_n of H_n extends to a double covering of the wreath product $H\Gamma_n := (\Gamma \times \mathbb{Z}_2)^n \rtimes S_n$ by $\tilde{H}\Gamma_n$:

$$1 \longrightarrow \mathbb{Z}_2 \xrightarrow{i} \tilde{H}\Gamma_n \xrightarrow{\theta_n} H\Gamma_n \longrightarrow 1.$$

The order $|\tilde{H}_n|$ is clearly $2^{n+1}n!|\Gamma|^n$, where $|\Gamma|$ denotes the order of Γ . The group $\tilde{H}\Gamma_n$ contains several distinguished subgroups: S_n , \tilde{H}_n , Γ^n , $\tilde{H}\Gamma_n$, and the wreath product $\Gamma_n := \Gamma^n \rtimes S_n$ etc.

We define a \mathbb{Z}_2 -grading on the group $\tilde{H}\Gamma_n$ by setting the degree of a_i to be 1 and the degree of elements in S_n and Γ^n to be 0, and denote by $\tilde{H}\Gamma_n^0$ (resp. $\tilde{H}\Gamma_n^1$) the degree zero (resp. one) part. This induces an epimorphism p from $\tilde{H}\Gamma_n$ or the group algebra $\mathbb{C}[\tilde{H}\Gamma_n]$ to \mathbb{Z}_2 . The homomorphism p descends to a homomorphism on $H\Gamma_n$ which will be denoted by p again. We say $x \in \tilde{H}\Gamma_n$ or $H\Gamma_n$ *even* (resp. *odd*) if $p(x)$ is 0 (resp. 1).

1.2. Partition-valued functions. A partition λ of a non-negative integer n is a monotonic non-increasing sequence of integers λ_i called parts such that $n = \lambda_1 + \cdots + \lambda_l = |\lambda|$. Here $l = l(\lambda)$ is the *length* of λ . We may also write $\lambda = (1^{m_1}2^{m_2}\cdots)$, where m_i is the number of times that i appears in λ . For two partitions λ and μ the dominance order $\lambda \geq \mu$ is defined by $\lambda_1 \geq \mu_1$, $\lambda_1 + \lambda_2 \geq \mu_1 + \mu_2$, etc. A partition λ is called *strict* if its parts are distinct integers.

We will use partitions indexed by Γ_* and Γ^* . For a finite set X and $\rho = (\rho(x))_{x \in X}$ a family of partitions indexed by X , we write $\|\rho\| = \sum_{x \in X} |\rho(x)|$. It is convenient to regard $\rho = (\rho(x))_{x \in X}$ as a partition-valued function on X . We denote by $\mathcal{P}(X)$ the set of all partitions indexed by X and by $\mathcal{P}_n(X)$ the subset consisting of ρ such that $\|\rho\| = n$. The total number of parts, denoted by $l(\rho) = \sum_x l(\rho(x))$, in the partition-valued function $\rho = (\rho(x))_{x \in X}$ is called the *length* of ρ . The *dominance order* on $\mathcal{P}(X)$ is defined naturally by $\rho \geq \pi$ if $\rho(x) \geq \pi(x)$ for each x . We also write $\rho \gg \pi$ if $\rho(x) \geq \pi(x)$ and $\rho(x) \neq \pi(x)$ for each $x \in X$. For $\rho \in \mathcal{OP}(\Gamma_*)$ we define $\bar{\rho} \in \mathcal{OP}(\Gamma_*)$ by $\bar{\rho}(c) = \rho(c^{-1})$, where c^{-1} denotes the conjugacy class $\{g|g^{-1} \in c\}$.

Let $\mathcal{OP}(X)$ be the set of partition-valued functions $(\rho(x))_{x \in X}$ in $\mathcal{P}(X)$ such that all parts of $\rho(x)$ are odd integers for each x , and let $\mathcal{SP}(X)$ be the set of ρ such that each $\rho(x)$ is strict. When X consists of a single element, we will omit X and simply write \mathcal{P} for $\mathcal{P}(X)$, thus \mathcal{OP} or \mathcal{SP} will be used accordingly. A variant of Euler's theorem says that the number of strict partition-valued functions on a set X is equal to the number of partition-valued functions on X with odd integer parts.

We denote by

$$\begin{aligned} \mathcal{P}_n^+(X) &= \{\lambda \in \mathcal{P}_n(X) \mid l(\rho) \equiv 0 \pmod{2}\}, \\ \mathcal{P}_n^-(X) &= \{\lambda \in \mathcal{P}_n(X) \mid l(\rho) \equiv 1 \pmod{2}\}, \end{aligned}$$

and define $\mathcal{SP}_n^\pm(X) = \mathcal{P}_n^\pm(X) \cap \mathcal{SP}_n(X)$ for $i = 0, 1$.

We will also need another parity d on partition-valued functions. For a partition-valued function $\rho = (\rho(x))_{x \in X}$ we define

$$(1.2) \quad d(\rho) = \|\rho\| - l(\rho).$$

1.3. Conjugacy classes of $H\Gamma_n$. Let Γ be a finite group with $r + 1$ conjugacy classes. We denote by $\Gamma^* = \{\gamma_i\}_{i=0}^r$ the set of complex irreducible characters where γ_0 is the trivial character, and by $\Gamma_* = \{c^i\}$ the set of conjugacy classes where c^0 is the identity conjugacy class. Let $|c|$ be the order of the conjugacy class $c \in \Gamma_*$, and then $\zeta_c := |\Gamma|/|c|$ is the order of the centralizer of an element in the class c .

For a subset $I = \{i_1, i_2, \dots, i_m\}$ of the set $\{1, \dots, n\}$, we denote $a_I = a_{i_1} a_{i_2} \cdots a_{i_m}$. It follows that $p(a_I) \equiv |I| \pmod{2}$. If $I \cap J = \emptyset$, then $a_I a_J = (-1)^{|I||J|} a_J a_I$. Also we can easily show by induction that

$$(1.3) \quad a_{i_1 i_2 \dots i_m} = z^{d(s)} a_{s(i_1) s(i_2) \dots s(i_m)}$$

for a permutation s of $I = \{i_1, i_2, \dots, i_m\}$. Clearly the morphism $\theta_n : \widetilde{H}_n \rightarrow H_n$ sends a_I to $b_I = b_{i_1} \cdots b_{i_m}$, where b_i are the generators of \mathbb{Z}_2^n .

The conjugacy classes of a wreath product is well understood, cf. [M, Z]. In particular this gives us the following description of conjugacy classes of the wreath product $H\Gamma_n = (\Gamma \times \mathbb{Z}_2)^n \rtimes S_n$. Given a cycle $t = (i_1 \dots i_m)$, we call the set $\{i_1, \dots, i_m\}$ the *support* of t , denoted by $\text{supp}(t)$. Given $(g, b_I s) \in H\Gamma_n$, every element $b_I s \in H_n$ can be uniquely written as a product (up to order)

$$b_I s = (b_{I_1} s_1) \cdots (b_{I_k} s_k),$$

where $s \in S_n$ is a product of disjoint cycles $s_1 \dots s_k$, and $b_{I_a} \in \mathbb{Z}_2^n$ so that $I_a \subset \text{supp}(s_a)$, and we call $b_{I_a} s_a$ a *signed cycle* of $b_I s$ with the sign $(-1)^{|I_k|}$. For each signed cycle $b_{I_k} s_k$ with $s_k = (j_1 \cdots j_m)$, the *signed cycle-product* of $b_{I_k} s_k$ is the element $g_{j_m} g_{j_{m-1}} \cdots g_{j_1}$ with the sign $(-1)^{|I_k|}$. For $c \in \Gamma_*$ let $m_i^+(c)$ (resp. $m_i^-(c)$) be the number of cycles of x whose signed cycle-product lies in c and has $+$ (resp. $-$) sign. Then (ρ^+, ρ^-) with $\rho^+(c) = (i^{m_i^+(c)})_{i \geq 1}$ and $\rho^-(c) = (i^{m_i^-(c)})$ is a pair of partition-valued functions on Γ_* such that $\|\rho^+\| + \|\rho^-\| = n$, and will be called the *type* of the element $(g, b_I s)$. Two elements of $H\Gamma_n$ are conjugate if and only if their types are the same. We say that the conjugacy class C_{ρ^+, ρ^-} is *even* (resp. *odd*) if it consists of even (resp. odd) elements. More precisely if $(g, b_I s) \in C_{\rho^+, \rho^-}$, then C_{ρ^+, ρ^-} is even (resp. odd) if $|I|$ is even (resp. odd).

1.4. Split conjugacy classes in $\tilde{H}\Gamma_n$. We can write a general element of $\tilde{H}\Gamma_n$ as $(g, z^k a_I s)$ where

$$z^k a_I s = z^k (a_{I_1} s_1) \cdots (a_{I_q} s_q),$$

and $s = s_1 \cdots s_q$ is a cycle decomposition of s and $I_j \subset \text{supp}(s_j)$. We denote by $J^c = \{1, \dots, n\} - J$ the complement of a subset $J \subset \{1, \dots, n\}$.

Lemma 1.1. *Let $a_I s = (a_{I_1} s_1) \cdots (a_{I_q} s_q)$ be an element of \tilde{H}_n in its cycle decomposition. Let $J = \text{supp}(s_1) \cap I_1^c$, then*

$$(a_J s_1)(a_I s)(a_J s_1)^{-1} = z^{d(s_1) + |I||J|} a_I s.$$

Consequently $(a_{I^c} s)(a_I s)(a_{I^c} s)^{-1} = z^{d(s) + |I||J|} a_I s$.

Proof. Observe that $a_I^2 = z^{(|I|+1)|I|/2}$ for any subset I . For $k \neq 1$ we have $(a_J s_1)(a_{I_k} s_k)(a_J s_1)^{-1} = z^{|J||I_k|} a_{I_k} s_k$. Therefore it reduces to see that

$$\begin{aligned} (a_J s_1)(a_{I_1} s_1)(a_J s_1)^{-1} &= z^{(|J|+1)|J|/2} a_J a_{s_1(I_1, J)} s_1 \\ &= z^{(|J|+1)|J|/2 + d(s_1)} a_J a_{(I_1, J)} s_1 \\ &= z^{(|J|+1)|J|/2 + d(s_1) + |J||I_1|} a_J^2 a_{I_1} s_1 \\ &= z^{|J||I_1| + d(s_1)} a_I s_1, \end{aligned}$$

where we have used the fact that $\text{supp}(s_1) = I_1 \cup J$. \square

If two elements of $\tilde{H}\Gamma_n$ are conjugate, then clearly their images are conjugate in $H\Gamma_n$. On the other hand, for any conjugacy class C of $\tilde{H}\Gamma_n$, $\theta_n^{-1}(C)$ is either a conjugacy class of $\tilde{H}\Gamma_n$ or it splits into two conjugacy classes of $\tilde{H}\Gamma_n$ (Indeed this holds in a more general setup, cf. e.g. [Jo1]). A conjugacy class C_{ρ^+, ρ^-} of $H\Gamma_n$ is called *split* if the preimage $\theta^{-1}(C_{\rho^+, \rho^-})$ splits into two conjugacy classes in $\tilde{H}\Gamma_n$. Equivalently, an element $x \in \tilde{H}\Gamma_n$ is called *split* if x is not conjugate to zx in $\tilde{H}\Gamma_n$, then C_{ρ^+, ρ^-} is split if and only if $\theta^{-1}(C_{\rho^+, \rho^-})$ consists of split elements. A conjugacy class of $\tilde{H}\Gamma_n$ *splits* if it consists of split elements.

The following theorem in the case when $\Gamma = 1$ was known in literature (cf. [Sg, St, Jo2]). It was also stated in [R] for Γ cyclic.

Theorem 1.2. *The conjugacy class C_{ρ^+, ρ^-} in $H\Gamma_n$ splits if and only if*

- (1) For even C_{ρ^+, ρ^-} , we have $\rho^+ \in \mathcal{OP}_n(\Gamma_*)$ and $\rho^- = \emptyset$,
- (2) For odd C_{ρ^+, ρ^-} , we have $\rho^+ = \emptyset$ and $\rho^- \in \mathcal{SP}_n^-(\Gamma_*)$.

Proof. For $(g, a_I s), (h, a_J t) \in \tilde{H}\Gamma_n$, it follows by definition that

$$(1.4) \quad \begin{aligned} & (h, a_J t)(g, a_I s)(h, a_J t)^{-1} \\ &= (ht(g)(tst^{-1})(h^{-1}), z^{(|J|+1)|J|/2} a_J a_{t(I)} a_{tst^{-1}(J)} tst^{-1}). \end{aligned}$$

(\implies) i) The conjugacy class C_{ρ^+, ρ^-} is even and split. Suppose on the contrary there is a part of even integer in $\rho^+(c)$. Without loss of generality we can assume that $\theta^{-1}(C_{\rho^+, \rho^-})$ contains a representative element $(g, a_I s)$ with the signed cycle decomposition

$$a_I s = (12 \cdots r)(a_{I_2} s_2) \cdots (a_{I_p} s_p),$$

where $r = 2k$ is even and I_1 is empty (we can take all I_i empty corresponding to parts in ρ^+) and $|I|$ is even. Consider the element

$$(h, a_J t) = (h, a_{12 \cdots r}(12 \cdots r)) \in \tilde{H}\Gamma_n,$$

where $h = (h_1, \dots, h_n)$ with $h_j = g_j$, for $j = 1, \dots, r$ and $h_j = 1$ otherwise. We claim that

$$(h, a_J t)(g, a_I s)(h, a_J t)^{-1} = z(g, a_I s).$$

In fact the j th component of $ht(g)s(h^{-1})$ equals $g_j g_{j-1} g_{j-1}^{-1} = g_j$ for $1 \leq j \leq r$ and it also equals $1 \cdot g_j \cdot 1 = g_j$ for $j \neq 1, \dots, r$. Noting that $st = ts$ we have

$$ht(g)(tst^{-1})(h^{-1}) = ht(g)s(h^{-1}) = g.$$

Moreover by Lemma 1.1 we have (recall $r = 2k$)

$$(a_J t)(a_I s)(a_J t)^{-1} = z^{(2k-1)+2k \cdot |I|} a_I s = z a_I s.$$

Thus $(g, a_I s)$ is conjugate to $z(g, a_I s)$ in view of Eqn. (1.4). Therefore if the even-parity conjugacy class C_{ρ^+, ρ^-} splits then $\rho^+ \in \mathcal{OP}(\Gamma_*)$.

Now suppose that $\rho^- \neq \emptyset$. Then ρ^- contains at least two parts since we assume that C_{ρ^+, ρ^-} is even. Without loss of generality we can assume that $\theta^{-1}(C_{\rho^+, \rho^-})$ contains an element $(g, a_I s)$ such that

$$a_I s = (a_{i_1} s_1)(a_{i_2} s_2)(a_{i_3} s_3) \cdots (a_{I_p} s_p),$$

where $i_1 \in \text{supp}(s_1), i_2 \in \text{supp}(s_2)$. If both s_1 and s_2 are of cycle length 1, then $(i_1 i_2)(g, a_I s)(i_1 i_2)^{-1} = z(g, a_I s)$. Assume that $\text{ord}(s_1) \geq 2$, so $s_1^{-1}(i_1) = i'_1 \neq i_1$. Consider $(h, a_{i_1} s_1)$, where $h_j = g_j$ for $j \in \text{supp}(s_1)$ and $h_j = 1$ otherwise. Then

$$\begin{aligned}
& (a_{i_1}s_1)^{-1}a_Is(a_{i_1}s_1) \\
&= (a_{i_1}s_1)^{-1}(a_{i_1}s_1a_{i_2}s_2)(a_{i_1}s_1)(a_{I_3}s_3) \cdots (a_{I_p}s_p) \\
&= a_{i_2}s_2(a_{i_1}s_1)(a_{I_3}s_3) \cdots (a_{I_p}s_p) \\
&= z(a_{i_1}s_1)(a_{i_2}s_2)(a_{I_3}s_3) \cdots (a_{I_p}s_p) \\
&= za_Is.
\end{aligned}$$

Subsequently $(h, a_{i_1}s_1)^{-1}(g, a_Is)(h, a_{i_1}s_1) = z(g, a_Is)$. Hence if C_{ρ^+, ρ^-} of even parity splits then ρ^- is empty. Together with the above we have shown that split conjugacy class of even parity should have property (1).

ii) The conjugacy class C_{ρ^+, ρ^-} is odd and split. If on the contrary $\rho^+ \neq \emptyset$, we can assume that $\theta^{-1}(C_{\rho^+, \rho^-})$ contains an element (g, a_Is) with the signed cycle decomposition

$$a_Is = (s_1)(a_{I_2}s_2) \cdots (a_{I_q}s_q),$$

where I_1 is empty and $|I|$ is odd. Take the element (h, a_Js_1) where $J = \text{supp}(s_1)$ and $h_j = g_j$ for $j \in J$ and $h_j = 1$ otherwise. Similarly we can verify that $(h, a_Js_1)(g, a_Is)(h, a_Js_1)^{-1} = z(g, a_Is)$ by using Lemma 1.1. In fact

$$(a_Js_1)(a_Is)(a_Js_1)^{-1} = z^{(|J|-1)+|I||J|}a_Is = za_Is,$$

since $|I|$ is odd. Hence $\theta^{-1}(C_{\rho^+, \rho^-})$ does not split if C_{ρ^+, ρ^-} is odd and $\rho^+ \neq \emptyset$.

Next we assume on the contrary that ρ^- contains two identical parts, then by taking conjugation if necessary we can assume that $\theta^{-1}(C_{\rho^+, \rho^-})$ contains an element (g, a_Is) such that

$$a_Is = (a_{i_1}(i_1i_2 \cdots i_k))(a_{j_1}(j_1j_2 \cdots j_k)) \cdots (a_{I_q}s_q)$$

and $(g_{i_1}, \dots, g_{i_k}) = (g_{i_1}, \dots, g_{i_k}) = (x, 1, \dots, 1)$ for some $x \in c \in \Gamma_*$. Consider the element $(1, t)$, where $t = (i_1j_1) \cdots (i_kj_k)$. Then we have

$$\begin{aligned}
(1, t)(g, a_Is)(1, t)^{-1} &= (t(g), a_{t(I)}s) \\
&= (g, a_{j_1}(j_1 \cdots j_k)a_{i_1}(i_1 \cdots i_k) \cdots) \\
&= (g, za_{i_1}(i_1 \cdots i_k)a_{j_1}(j_1 \cdots j_k) \cdots) = (g, za_Is),
\end{aligned}$$

which is a contradiction.

(\Leftarrow) Suppose that (1) holds. If on the contrary the even conjugacy class $C_{\rho^+, \emptyset}$ does not split, then we can assume that $\theta^{-1}(C_{\rho^+, \emptyset})$ contains an element (g, s) , where $s = s_1 \cdots s_q$ with each s_i being odd cycle, and $(h, a_Jt)(g, s)(h, a_Jt)^{-1} = z(g, s)$. Therefore $(a_Jt)s(a_Jt)^{-1} = zs$, i.e., $za_{s(J)} = a_J$, which in particular implies that $\text{supp}(s) \subset J$. Then

$a_{s(J)} = z^{d(s)} a_J$ by Eq. (1.3), and so $d(s) = 1 \bmod 2$ which contradicts with the assumption on s .

Suppose that (2) holds. Assume on the contrary that the odd conjugacy class C_{\emptyset, ρ^-} does not split. Since an identification of two elements in $H\Gamma_n$ implies that their respective components in $H\Gamma_{|\rho^-(c)|}$ are already equal, we can assume that ρ^- consists of one strict partition $\rho(c)^-$ for some c . Thus $\theta^{-1}(C_{\emptyset, \rho^-})$ contains a non-split element $(g, a_I s) = (g, a_{i_1} s_1 \cdots a_{i_q} s_q)$, where q is odd and $i_k \in \text{supp}(s_k)$. Let $(h, a_J t)(g, a_I s)(h, a_J t)^{-1} = z(g, a_I s)$ for some element $(h, a_J t)$. It follows that t commutes with s and s is a product of disjoint cycles with mutually distinct orders, the permutation t equals $s_1^{r_1} s_2^{r_2} \cdots s_q^{r_q}$ for $0 \leq r_i \leq \text{ord}(s_i)$. Thus we can write $a_J t = (a_{J_1} t_1) \cdots (a_{J_q} t_q)$ with $t_k = s_k^{r_k}$. As in the proof of Lemma 1.1 we have

$$\begin{aligned} & (a_J t)(a_{i_1} s_1)(a_J t)^{-1} \\ &= (a_{J_1} t_1)(a_{J_2} t_2) \cdots (a_{J_q} t_q)(a_{i_1} s_1)((a_{J_2} t_2) \cdots (a_{J_q} t_q))^{-1}(a_{J_1} t_1)^{-1} \\ &= z^{|J|-|J_1|}(a_{J_1} t_1)(a_{i_1} s_1)(a_{J_1} t_1)^{-1}, \end{aligned}$$

which must equal $a_{i_1} s_1$ up to a power of z . Set $(a_{J_1} t_1)(a_{i_1} s_1)(a_{J_1} t_1)^{-1} = z^* a_{i_1} s_1$ where $*$ is 0 or 1. We claim that $*$ is always 0. Without loss of generality we let $J_1 = \{1, 2, \dots, r\}$, $i_1 = 1$, $s_1 = (12 \cdots k)$ with $0 \leq r \leq k$, then

$$a_{J_1} a_{s_1^{r_1}(1)} a_{s_1(J_1)} z^{(r+1)r/2} = a_{12 \cdots r} a_{r_1} a_{23 \cdots r+1} z^{(r+1)r/2} = z^* a_1$$

implies that $r_1 = r+1$, which in turn implies the exponent $*$ is equal to 0. Therefore $(a_J t)(a_{i_1} s_1)(a_J t)^{-1} = z^{|J|-|J_1|} a_{i_1} s_1$ and similarly we have

$$\begin{aligned} & (a_J t)(a_I s)(a_J t)^{-1} = (a_J t)(a_{I_1} s_1) \cdots (a_{I_q} s_q)(a_J t)^{-1} \\ &= z^{q|J|-(|J_1|+\cdots+|J_q|)} a_I s = z^{(q-1)|J|} a_I s = a_I s, \end{aligned}$$

since q is odd. This is a contradiction. \square

For $\rho \in \mathcal{OP}_n(\Gamma_*)$ we let $D_\rho = D_\rho^+$ be the split conjugacy class in $\tilde{H}\Gamma_n$ containing the elements (g, s) of type (ρ, \emptyset) with $g \in \Gamma^n$, $s \in S_n$. Then zD_ρ^+ is the other conjugacy class lying in $\theta_n^{-1}(C_{\rho, \emptyset})$, which will be denoted by D_ρ^- .

For each partition-valued function $\rho = (\rho(c))_{c \in \Gamma_*}$ we define

$$Z_\rho = \prod_{c \in \Gamma_*, i \geq 1} m_i(c)! i^{m_i(c)} \zeta_c^{l(\rho(c))},$$

which is the order of the centralizer of an element of conjugacy type $\rho = (\rho(c))_{c \in \Gamma_*}$ in Γ_n (cf. [M]). It follows from Theorem 1.2 that the

order of the centralizer of an element in the conjugacy class D_ρ^+ is

$$(1.5) \quad 2^{1+l(\rho)} Z_\rho.$$

2. SPIN SUPERMODULES OVER $\tilde{H}\Gamma_n$

In this section, we first recall some general facts about *spin* supermodules over a superalgebra. This will be applied to (the group algebra of) $\tilde{H}\Gamma_n$. We introduce the Grothendieck group $R^-(\tilde{H}\Gamma_n)$ of spin supermodules over $\tilde{H}\Gamma_n$, and construct the so-called basic spin supermodules of $\tilde{H}\Gamma_n$. We further define a natural Hopf algebra structure on $R_{H\Gamma}^- = \bigoplus_{n \geq 0} R^-(\tilde{H}\Gamma_n)$, and introduce a weighted bilinear form on $R_{H\Gamma}^-$ associated to any given self-dual virtual character ξ of Γ . When ξ is trivial, the weighted bilinear form reduces to the standard one.

2.1. Superalgebras and supermodules. A complex *superalgebra* $A = A_0 \oplus A_1$ is a \mathbb{Z}_2 -graded complex vector space with a binary product $A \times A \longrightarrow A$ such that $A_i A_j \subset A_{i+j}$. A \mathbb{Z}_2 -graded vector space $V = V_0 \oplus V_1$ is a *supermodule* for a superalgebra A if $A_i V_j \subset V_{i+j}$. A linear map $f : M \rightarrow N$ between two A -supermodules is a morphism of degree i if $f(M_j) \subset M_{i+j}$ and for any homogeneous element $a \in A$ and any homogeneous vector $m \in M$ we have

$$f(am) = (-1)^{p(a)p(f)} a f(m).$$

Let $V = V_0 \oplus V_1$ and $W = W_0 \oplus W_1$ be two supermodules. The tensor product $V \otimes W$ is also a supermodule with $(V \otimes W)_i = \sum_{k+l=i \pmod{2}} V_k \otimes W_l$. The notions of submodules, irreducible supermodules etc are defined similarly as usual.

Let $\mathbb{C}^{r|s}$ be the \mathbb{Z}_2 -graded vector space $\mathbb{C}^r \oplus \mathbb{C}^s$. The algebra $M(r|s)$ consisting of all linear transformations on $\mathbb{C}^{r|s}$ inherits a natural \mathbb{Z}_2 grading from $\mathbb{C}^{r|s}$. It is easily seen that $M(r|s)$ is a simple superalgebra.

Another example of simple superalgebra is $Q(n)$, which is the subalgebra of $M(n|n)$ consisting of matrices of the form

$$\begin{bmatrix} C & D \\ D & C \end{bmatrix}, \quad C, D \in M_n(\mathbb{C}).$$

The left multiplication of $Q(n)$ on $\mathbb{C}^{n|n}$ gives an irreducible supermodule structure on it.

A well-known result due to C. T. C. Wall says that these superalgebras are the only simple superalgebras over \mathbb{C} . In the sequel we will say the supermodules $\mathbb{C}^{r|s}$ and $\mathbb{C}^{n|n}$ are of type M and Q respectively.

Let G be a finite group and let $p : G \rightarrow \mathbb{Z}_2$ be a group epimorphism. We denote by G_0 the kernel of p which is a subgroup of G of index 2. We regard $p(\cdot)$ as a parity function on G by letting the degree of elements in G_0 be 0 and letting the degree of elements in the complementary $G_1 = G \setminus G_0$ be 1. Elements in G_0 (resp. G_1) will be called even (resp. odd). In addition we assume that G contains a distinguished even central element z of order 2. A *spin* supermodule over G is a supermodule over the group superalgebra $\mathbb{C}[G]$ such that z acts as -1 . The group superalgebra is semisimple (cf. [Jo1]), i.e. decomposes into a direct sum of simple superalgebras. We will refer the corresponding supermodules as of type M and type Q .

Now let us return to our main example $\tilde{H}\Gamma_n$. It is easy to see that the characters of spin supermodules vanish on nonsplit classes. Let $(-1)^p$ be the one-dimensional representation of $\tilde{H}\Gamma_n$ given by $x \mapsto (-1)^{p(x)}$. A representation π of $\tilde{H}\Gamma_n$ is called a *double spin* representation if $(-1)^p \pi \simeq \pi$. If $\pi' = (-1)^p \pi \neq \pi$, then π' and π are called *associate spin* representations of $\tilde{H}\Gamma_n$. By the general theory of supermodules (cf. [Jo1]) and Theorem 1.2 we obtain the following proposition.

Proposition 2.1. *The number of irreducible double spin representations over $\tilde{H}\Gamma_n$ is equal to $|\mathcal{SP}_n^+(\Gamma_*)|$, and the number of pairs of irreducible associate spin representations is $|\mathcal{SP}_n^-(\Gamma_*)|$. The number of irreducible spin supermodules of Γ_n is $|\mathcal{SP}_n(\Gamma_*)|$.*

If V is an irreducible $\tilde{H}\Gamma_n$ -supermodule of type M , then its underlying module $|V|$ (by forgetting the \mathbb{Z}_2 -grading structure) is an irreducible double spin $\tilde{H}\Gamma_n$ -module. If P is an irreducible $\tilde{H}\Gamma_n$ -supermodule of type Q , then $|P| \simeq N \oplus N'$ where N and N' are a pair of irreducible associate spin $\tilde{H}\Gamma_n$ -modules.

Remark 2.1. It has been observed (cf. [St, Jo2, Naz]) that there is a very close connection between the representations of \tilde{H}_n and those of a spin cover \tilde{S}_n of the symmetric group S_n . Yamaguchi [Y] explains such an phenomenon in an elegant way by establishing an isomorphism between the group superalgebra $\mathbb{C}[\tilde{H}_n]/\langle z = -1 \rangle$ and the (outer) tensor product of the group superalgebra $\mathbb{C}[\tilde{S}_n]/\langle z = -1 \rangle$ with the complex Clifford algebra of n variables. (Note that a Clifford algebra admits a unique irreducible supermodule). In view of this, there is also an isomorphism by substituting \tilde{H}_n and \tilde{S}_n with $\tilde{H}\Gamma_n$ and $\tilde{\Gamma}_n$ respectively. This isomorphism provides a direct connection between the constructions in the present paper and [FJW2].

Given a \mathbb{Z}_2 -graded finite group G and a \mathbb{Z}_2 -graded subgroup H that contain an even central element z of order 2, we can define the induction and restriction of supermodules similarly as usual. In particular the induced supermodule (or a restriction) of a spin supermodule remains to be a spin supermodule. The Mackey theorem remains true in this setup (cf. [FJW2], Sect. 3).

2.2. The Grothendieck group $R^-(\tilde{H}\Gamma_n)$. Let $R(\Gamma) = \oplus_{i=0}^r \mathbb{C}\gamma_i$ be the space of complex-valued class functions on Γ . The usual bilinear form on $R(\Gamma)$ is defined as follows:

$$\langle f, g \rangle_\Gamma = \frac{1}{|\Gamma|} \sum_{x \in \Gamma} f(x)g(x^{-1}) = \sum_{c \in \Gamma_*} \zeta_c^{-1} f(c)g(c^{-1}).$$

Then $R_{\mathbb{Z}}(\Gamma) := \oplus_{i=0}^r \mathbb{Z}\gamma_i$ becomes an integral lattice in $R(\Gamma)$ under the standard bilinear form: $\langle \gamma_i, \gamma_j \rangle = \delta_{ij}$.

A *spin class function* on $\tilde{H}\Gamma_n$ is class function $f : \tilde{H}\Gamma_n \rightarrow \mathbb{C}$ such that $f(zx) = -f(x)$, hence it vanishes on non-split conjugacy classes. A *spin super class function* f on $\tilde{H}\Gamma_n$ is a spin class function such that it vanishes further on odd conjugacy classes. In other words, f corresponds to a complex functional on $\mathcal{OP}_n(\Gamma_*)$ in view of Theorem 1.2.

Let $R^-(\tilde{H}\Gamma_n)$ be the \mathbb{C} -span of spin super class functions on $\tilde{H}\Gamma_n$. The standard bilinear form on $R^-(\tilde{H}\Gamma_n)$ is given by

$$\langle f, g \rangle_{\tilde{H}\Gamma_n} = \frac{1}{|\tilde{H}\Gamma_n|} \sum_{x \in \tilde{H}\Gamma_n} f(x)g(x^{-1}) = \sum_{\rho \in \mathcal{OP}_n(\Gamma_*)} \frac{1}{2^{l(\rho)} Z_\rho} f(\rho)g(\bar{\rho}),$$

where $f, g \in R^-(\tilde{H}\Gamma_n)$, $f(\rho) = f(D_\rho^+)$, and Eqn. (1.5) is used here. The following holds for a general \mathbb{Z}_2 -graded finite group [Jo1].

Proposition 2.2. *The characters of irreducible spin supermodules over $\tilde{H}\Gamma_n$ form a \mathbb{Z} -basis of $R^-(\tilde{H}\Gamma_n)$. Let ϕ and γ be two irreducible characters of spin supermodules, then*

$$(2.1) \quad \langle \phi, \gamma \rangle = \begin{cases} 1 & \text{if } \phi \simeq \gamma, \text{ type } M \\ 2 & \text{if } \phi \simeq \gamma, \text{ type } Q \\ 0 & \text{otherwise} \end{cases}.$$

Conversely, if $\langle f, f \rangle = 1$ for $f \in R^-(\tilde{H}\Gamma_n)$, then f or $-f$ affords an irreducible spin $\tilde{H}\Gamma_n$ -supermodule of type M .

2.3. Basic spin supermodules. Let L_n be the Clifford algebra generated by e_1, e_2, \dots, e_n with relations:

$$(2.2) \quad \{e_i, e_j\} = e_i e_j + e_j e_i = -2\delta_{ij}.$$

The group \tilde{H}_n acts on L_n by $a_i \cdot e_I = e_i e_I$ and $s \cdot e_i = e_{s(i)}$.

The Clifford algebra L_n has a natural \mathbb{Z}_2 -grading given by the parity p such that $p(e_i) = 1$. The set $\{a_I\}, I \subset \{1, \dots, n\}$ form a basis of L_n . We observe that $s(a_I) = (-1)^{d(s)} a_I$ ($s \in S_n$) if and only if I is a union of $\text{supp}(s_i)$, where $s = s_1 \cdots s_l$ is a cycle decomposition.

Proposition 2.3. [Jo2] *The module L_n is an irreducible \tilde{H}_n -supermodule. The value of its character χ_n at the conjugacy class D_α^+ is given by*

$$(2.3) \quad \chi_n(\alpha) = 2^{l(\alpha)}, \quad \alpha \in \mathcal{OP}_n.$$

Let V be a Γ -module afforded by the character $\gamma \in R(\Gamma)$, the tensor product $V^{\otimes n}$ is a Γ_n -module by the direct product action of Γ^n combined with permutation action of the symmetric group S_n . More explicitly the action is given by

$$(g, s) \cdot (v_i \otimes \cdots \otimes v_n) = g_1 v_{s^{-1}(1)} \otimes \cdots \otimes g_n v_{s^{-1}(n)},$$

where $g = (g_1, \dots, g_n) \in \Gamma^n, s \in S_n$. We denote the resulting character by $\eta_n(\gamma)$. Indeed one can extend (cf. [W1, FJW1]) this construction to define a map $\eta_n : R(\Gamma) \rightarrow R(\Gamma_n)$.

Let V be a Γ -module afforded by the character $\gamma \in R(\Gamma)$, and let U be a spin supermodule of \tilde{H}_n with the character π_n . The tensor product $V^{\otimes n} \otimes U$ has a canonical spin supermodule structure for $\tilde{H}\Gamma_n$ as follows. For any $g = (g_1, \dots, g_n) \in \Gamma^n$, and an element (g, a_{Is}) in $\tilde{H}\Gamma_n$, the action is given by

$$(g, a_{Is}) \cdot (v_i \otimes \cdots \otimes v_n \otimes u) = g_1 v_{s^{-1}(1)} \otimes \cdots \otimes g_n v_{s^{-1}(n)} \otimes (a_{Is} u).$$

It is easy to see that if V and U are irreducible, then so is $V^{\otimes n} \otimes U$. We denote by $\pi_n(\gamma)$ the character of this spin supermodule. The following result can be proved as in [FJW1, FJW2] for similar results.

Proposition 2.4. *Let π_n be the character of a spin \tilde{H}_n -supermodule. Then the value of the character $\pi_n(\gamma)$ at an element $(g, a_{Is}) \in D_\rho^+$ in the even split conjugacy class D_ρ^+ is given by*

$$(2.4) \quad \pi_n(\gamma)((g, a_{Is})) = \pi_n(a_{Is}) \prod_{c \in \Gamma_*} \gamma(c)^{l(\rho(c))}.$$

The map π_n can be extended $R(\Gamma)$ to $R^-(\tilde{H}\Gamma_n)$ (compare [FJW2]):

$$(2.5) \quad \pi_n(\beta - \gamma) = \sum_{m=0}^n (-1)^m \text{Ind}_{\tilde{H}\Gamma_{n-m} \hat{\times} \tilde{H}\Gamma_m}^{\tilde{H}\Gamma_n} [\pi_{n-m}(\beta) \otimes \pi_m(\gamma)].$$

In particular, when U is the spin \tilde{H}_n -supermodule L_n we denote by $\chi_n(\gamma)$ the character of the $\tilde{H}\Gamma_n$ -supermodule $L_n(V) := V^{\otimes n} \otimes L_n$. We will refer to $L_n(V)$ (associated to irreducible V) as the *basic* spin supermodules over $\tilde{H}\Gamma_n$.

Corollary 2.2. *The character values of $\chi_n(\gamma)$ on the conjugacy classes D_ρ^\pm are given by*

$$(2.6) \quad \chi_n(\gamma)(D_\rho^\pm) = \pm 2^{l(\rho)} \prod_{c \in \Gamma_*} \gamma(c)^{l(\rho(c))}, \quad \rho \in \mathcal{OP}_n(\Gamma_*).$$

2.4. Hopf algebra structure on $R_{H\Gamma}^-$. Our main object of this paper is the space

$$R_{H\Gamma}^- = \bigoplus_{n \geq 0} R^-(\tilde{H}\Gamma_n).$$

Let $\tilde{H}\Gamma_n \hat{\times} \tilde{H}\Gamma_m$ be the direct product of $\tilde{H}\Gamma_n$ and $\tilde{H}\Gamma_m$ with a twisted multiplication $(t, t') \cdot (s, s') = (ts, z^{p(t')p(s)} t' s')$, where $s, t \in \tilde{H}\Gamma_n, s', t' \in \tilde{H}\Gamma_m$ are homogeneous. We define the *spin product* of $\tilde{H}\Gamma_n$ and $\tilde{H}\Gamma_m$ by

$$(2.7) \quad \tilde{H}\Gamma_n \hat{\times} \tilde{H}\Gamma_m = \tilde{H}\Gamma_n \hat{\times} \tilde{H}\Gamma_m / \{(1, 1), (z, z)\},$$

which can be embedded into the spin group $\tilde{H}\Gamma_{n+m}$ canonically by letting $(t'_i, 1) \mapsto t_i, (1, t''_j) \mapsto t_{n+j}, i = 1, \dots, n-1, j = 1, \dots, m-1$. We will identify $\tilde{H}\Gamma_n \hat{\times} \tilde{H}\Gamma_m$ with its image in $\tilde{H}\Gamma_{n+m}$. The subgroup $\tilde{H}\Gamma_n \hat{\times} \tilde{H}\Gamma_m$ has a distinguished subgroup of index 2 consisting of even elements with $p = 0$. Therefore we can define $R^-(\tilde{H}\Gamma_n \hat{\times} \tilde{H}\Gamma_m)$ to be the space of super spin class functions.

For two spin modules U and V of $\tilde{H}\Gamma_n$ and $\tilde{H}\Gamma_m$ we define the (*outer*)-*tensor product* by

$$(t, s) \cdot (u \otimes v) = (-1)^{p(s)p(u)} (tu \otimes sv),$$

where s and u are homogeneous elements. Since $(z, 1) \cdot (u \otimes v) = -u \otimes v$, the tensor $U \otimes V$ is a spin $\tilde{H}\Gamma_n \hat{\times} \tilde{H}\Gamma_m$ -supermodule. The following is a straightforward generalization of a result in [Jo2] for \tilde{H}_n .

Proposition 2.5. *Let U and V be simple supermodules for $\tilde{H}\Gamma_n$ and $\tilde{H}\Gamma_m$ respectively. Then*

- 1) If both U and V are of type M , then $U \otimes V$ is a simple $\tilde{H}\Gamma_n \hat{\times} \tilde{H}\Gamma_m$ -supermodule of type M .
- 2) If U and V are of different type, then $U \otimes V$ is a simple $\tilde{H}\Gamma_n \hat{\times} \tilde{H}\Gamma_m$ -supermodule of type Q .
- 3) If both U and V are type Q , then $U \otimes V \simeq N \oplus N$ for some simple $\tilde{H}\Gamma_n \hat{\times} \tilde{H}\Gamma_m$ -supermodule N of type M . We will denote N by $2^{-1}U \otimes V$.

For a simple supermodule V we define $c(V) = 0$ if V is type M and $c(V) = 1$ if V is type Q . Set $c(V_1, V_2) = c(V_1)c(V_2)$ for two simple supermodules of $\mathbb{C}[\tilde{H}\Gamma_n]$ and $\mathbb{C}[\tilde{H}\Gamma_m]$ respectively. It is easy to see that c satisfies the cocycle condition

$$(2.8) \quad c(V_1, V_2) + c(V_1 \otimes V_2, V_3) = c(V_2, V_3) + c(V_1, V_2 \otimes V_3).$$

It follows from Proposition 2.5 and Eqn. (2.8) that the tensor product defines an isomorphism:

$$R^-(\tilde{H}\Gamma_n) \bigotimes R^-(\tilde{H}\Gamma_m) \xrightarrow{\phi_{n,m}} R^-(\tilde{H}\Gamma_n \hat{\times} \tilde{H}\Gamma_m),$$

where $\phi_{n,m}(V_1 \otimes V_2) = 2^{-c(V_1, V_2)} V_1 \otimes V_2$ for simple modules V_1 and V_2 .

We now define a multiplication on $R_{H\Gamma}^-$ by the composition

$$(2.9) \quad m : R^-(\tilde{H}\Gamma_n) \bigotimes R^-(\tilde{H}\Gamma_m) \xrightarrow{\phi_{n,m}} R^-(\tilde{H}\Gamma_n \hat{\times} \tilde{H}\Gamma_m) \xrightarrow{Ind} R^-(\tilde{H}\Gamma_{n+m}),$$

and a comultiplication on $R_{H\Gamma}^-$ by the composition

$$(2.10) \quad \Delta : R^-(\tilde{H}\Gamma_n) \xrightarrow{Res} \bigoplus_{m=0}^n R^-(\tilde{H}\Gamma_{n-m} \times \tilde{H}\Gamma_m) \\ \xrightarrow{\phi^{-1}} \bigoplus_{m=0}^n R^-(\tilde{H}\Gamma_{n-m}) \bigotimes R^-(\tilde{H}\Gamma_m).$$

Here Ind and Res denote the induction and restriction of spin supermodules respectively, and the isomorphism ϕ^{-1} is given by $\bigoplus_{m=0}^n \phi_{n-m,m}^{-1}$.

Theorem 2.3. *The above operations define a Hopf algebra structure for $R_{H\Gamma}^-$.*

Proof. The associativity follows from the observation that the two different embeddings are conjugate:

$$(\tilde{H}\Gamma_n \hat{\times} \tilde{H}\Gamma_m) \hat{\times} \tilde{H}\Gamma_l \hookrightarrow \tilde{H}\Gamma_{n+m+l} \hookleftarrow \tilde{H}\Gamma_n \hat{\times} (\tilde{H}\Gamma_m \hat{\times} \tilde{H}\Gamma_l).$$

Using the cocycle condition (2.8) we can check the coassociativity as we did in [FJW2]. Using the cocycle c again and super analog of Mackey's

theorem we can check that Δ preserves the multiplication structure, for details see [FJW2] in a similar situation (also compare [Z, W1] for Hopf algebra structures in different but related setups). \square

2.5. A weighted bilinear form on $R(\Gamma)$ and $R^-(\tilde{H}\Gamma_n)$. Let ξ be a self-dual virtual character in $R_{\mathbb{Z}}(\Gamma)$, i.e. $\xi(c) = \xi(c^{-1})$, the *weighted* bilinear form on $R_{\mathbb{Z}}(\Gamma)$ (cf. [FJW1]) is defined by

$$\langle f, g \rangle_{\xi} = \langle \xi * f, g \rangle_{\Gamma}, \quad f, g \in R_{\mathbb{Z}}(\Gamma),$$

where $*$ the product of two characters. The self-duality of ξ is equivalent to the condition that the matrix of the weighted bilinear form is symmetric. When ξ is the trivial character g_0 of Γ , the weighted bilinear form becomes the standard one. One extends the weighted bilinear form to $R(\Gamma)$ by bilinearity.

Let $V = V_0 + V_1$ be a spin supermodule for $\tilde{H}\Gamma_n$ and W a module for Γ_n . The tensor product $W \otimes V = W \otimes V_0 \oplus W \otimes V_1$ carries a natural \mathbb{Z}_2 -grading and admits a natural spin $\tilde{H}\Gamma_n$ -supermodule structure by letting

$$(2.11) \quad (g, a_I \sigma)(w \otimes v) = (g, \sigma) \cdot w \otimes (g, a_I \sigma) \cdot v,$$

where $g \in \Gamma^n$, $a_I \in \Pi_n$, $\sigma \in S_n$. This gives rise to a morphism:

$$(2.12) \quad R(\Gamma_n) \otimes R^-(\tilde{H}\Gamma_n) \xrightarrow{\otimes} R^-(\tilde{H}\Gamma_n).$$

Recall we have defined $\eta_n(\xi) \in R(\Gamma_n)$ associated to $\xi \in R(\Gamma)$. Its character value at the class $\rho = (\rho(c))_{c \in \Gamma_*}$ is given by (cf. [FJW1, M])

$$(2.13) \quad \eta_n(\xi)(\rho) = \prod_{c \in \Gamma_*} \xi(c)^{l(\rho(c))}.$$

Thus $\eta_n(\xi)$ is self-dual as long as ξ is.

We now introduce a *weighted bilinear form* on $R^-(\tilde{H}\Gamma_n)$ by letting

$$(2.14) \quad \begin{aligned} \langle f, g \rangle_{\xi} &= \langle \eta_n(\xi) \otimes f, g \rangle_{\tilde{H}\Gamma_n} \\ &= \sum_{\rho \in \mathcal{OP}_n(\Gamma^*)} \frac{1}{2^{l(\rho)} Z_{\rho}} f(\rho) g(\bar{\rho}) \prod_{c \in \Gamma_*} \xi(c)^{l(\rho(c))}, \end{aligned}$$

where $f, g \in R^-(\tilde{H}\Gamma_n)$. The self-duality of $\eta_n(\xi)$ implies that the bilinear form $\langle \cdot, \cdot \rangle_{\xi}$ is symmetric. When ξ is taken to be the trivial character γ_0 , then it reduces to the standard bilinear form on $R^-(\tilde{H}\Gamma_n)$.

The bilinear form on $R_{H\Gamma}^- = \bigoplus_n R^-(\tilde{H}\Gamma_n)$ is given by

$$\langle u, v \rangle_{\xi} = \sum_{n \geq 0} \langle u_n, v_n \rangle_{\xi, \tilde{H}\Gamma_n},$$

where $u = \sum_n u_n$ and $v = \sum_n v_n$ with $u_n, v_n \in R^-(\tilde{H}\Gamma_n)$.

Remark 2.4. When Γ is finite subgroup of $SL_2(\mathbb{C})$, an importance choice for ξ is $\xi = 2\gamma_0 - \pi$, where π is the character afforded by the 2-dimensional representation of Γ given by the embedding of Γ in $SL_2(\mathbb{C})$. We shall see that the weighted bilinear form $\langle \cdot, \cdot \rangle_\xi$ on $R_{H\Gamma}^-$ becomes positive semi-definite. This will play an important role in the later part of this paper.

3. IDENTIFICATION OF $R_{H\Gamma}^-$ AS A FOCK SPACE

In this section, we first recall a twisted Heisenberg algebra $\widehat{\mathfrak{h}}_{\Gamma,\xi}[-1]$ and its Fock space $S_{H\Gamma}^-$ together with a bilinear form. We define an action of $\widehat{\mathfrak{h}}_{\Gamma,\xi}[-1]$ on $R_{H\Gamma}^-$ in terms of group-theoretic maps. We further show that there is a natural isometric isomorphism from $R_{H\Gamma}^-$ to $S_{H\Gamma}^-$ which is compatible with the Hopf algebra structure and Heisenberg algebra action on both spaces.

3.1. A twisted Heisenberg algebra $\widehat{\mathfrak{h}}_{\Gamma,\xi}[-1]$. Associated with a finite group Γ and a self-dual class function $\xi \in R(\Gamma)$, a twisted Heisenberg algebra $\widehat{\mathfrak{h}}_{\Gamma,\xi}[-1]$ (cf. [FJW2]) is generated by $a_m(\gamma), m \in 2\mathbb{Z} + 1, \gamma \in \Gamma^*$ and a central element C , subject to the relations:

$$(3.1) \quad [a_m(\gamma), a_n(\gamma')] = \frac{m}{2} \delta_{m,-n} \langle \gamma, \gamma' \rangle_\xi C, \quad m, n \in 2\mathbb{Z} + 1, \gamma, \gamma' \in \Gamma^*.$$

For $\gamma = \sum_{i=0}^r s_i \gamma_i \in R(\Gamma)$ ($s_i \in \mathbb{C}$) we write $a_m(\gamma) = \sum_i s_i a_m(\gamma_i)$. The center of $\widehat{\mathfrak{h}}_{\Gamma,\xi}[-1]$ is spanned by C together with $a_m(\gamma), m \in 2\mathbb{Z} + 1, \gamma \in R_0$, the radical of the bilinear form $\langle \cdot, \cdot \rangle_\xi$ in $R(\Gamma)$.

We introduce another basis for $\widehat{\mathfrak{h}}_{\Gamma,\xi}[-1]$:

$$(3.2) \quad a_m(c) = \sum_{\gamma \in \Gamma^*} \gamma(c^{-1}) a_m(\gamma), \quad m \in 2\mathbb{Z} + 1, c \in \Gamma_*.$$

Equivalently $a_m(\gamma) = \sum_{c \in \Gamma_*} \zeta_c^{-1} \gamma(c) a_m(c)$. Then the commutation relations (3.1) imply that

$$(3.3) \quad [a_m(c'^{-1}), a_n(c)] = \frac{m}{2} \delta_{m,-n} \delta_{c',c} \zeta_c \xi(c) C, \quad c, c' \in \Gamma_*,$$

where $m, n \in \mathbb{Z}$.

3.2. The Fock space $S_{H\Gamma}^-$. The Fock space $S_{H\Gamma}^-$ is defined to be the symmetric algebra generated by $a_{-n}(\gamma), n \in 2\mathbb{Z}_+ + 1, \gamma \in \Gamma^*$. There is a natural grading on $S_{H\Gamma}^-$ by letting

$$\deg(a_{-n}(\gamma)) = n, \quad n \in 2\mathbb{Z}_+ + 1,$$

which makes $S_{H\Gamma}^-$ into a \mathbb{Z}_+ -graded space.

We define an action of $\widehat{\mathfrak{h}}_{\Gamma,\xi}[-1]$ on $S_{H\Gamma}^-$ as follows: $a_{-n}(\gamma), n > 0$ acts as multiplication operator on $S_{H\Gamma}^-$ and C as the identity operator; $a_n(\gamma), n > 0$ acts as a derivation of algebra:

$$\begin{aligned} & a_n(\gamma) \cdot a_{-n_1}(\alpha_1) \cdots a_{-n_k}(\alpha_k) \\ &= \sum_{i=1}^k \delta_{n,n_i} \langle \gamma, \alpha_i \rangle_{\xi} a_{-n_1}(\alpha_1) \cdots \check{a}_{-n_i}(\alpha_i) \cdots a_{-n_k}(\alpha_k). \end{aligned}$$

Here $n_i > 0, \alpha_i \in R(\Gamma)$ for $i = 1, \dots, k$, and $\check{a}_{-n_i}(\alpha_i)$ means the very term is deleted. Note that $S_{H\Gamma}^-$ is not an irreducible representation over $\widehat{\mathfrak{h}}_{\Gamma,\xi}$ in general since the bilinear form $\langle \cdot, \cdot \rangle_{\xi}$ may be degenerate.

Denote by $S_{H\Gamma}^0$ the ideal in the symmetric algebra $S_{H\Gamma}^-$ generated by $a_{-n}(\gamma), n \in \mathbb{N}, \gamma \in R_0$. Denote by $\overline{S}_{H\Gamma}$ the quotient $S_{H\Gamma}^-/S_{H\Gamma}^0$. It follows from the definition that $S_{H\Gamma}^0$ is a submodule of $S_{H\Gamma}^-$ over the Heisenberg algebra $\widehat{\mathfrak{h}}_{\Gamma,\xi}[-1]$. In particular, this induces a Heisenberg algebra action on $\overline{S}_{H\Gamma}$ which is irreducible. The unit 1 in the symmetric algebra $S_{H\Gamma}^-$ is the highest weight vector. We will also denote by 1 its image in the quotient $\overline{S}_{H\Gamma}$.

3.3. The bilinear form on $S_{H\Gamma}^-$. The Fock space $S_{H\Gamma}^-$ admits a bilinear form $\langle \cdot, \cdot \rangle'_{\xi}$ determined by

$$(3.4) \quad \langle 1, 1 \rangle'_{\xi} = 1, \quad a_n(\gamma)^* = a_{-n}(\gamma), \quad n \in 2\mathbb{Z} + 1, n \in \Gamma^*.$$

Here $a_n(\gamma)^*$ denotes the adjoint of $a_n(\gamma)$.

For $\lambda \in \mathcal{OP}$ we write $a_{-\lambda}(\gamma) = a_{-\lambda_1}(\gamma) a_{-\lambda_2}(\gamma) \cdots$ for $\gamma \in \Gamma^*$, and $a_{-\lambda}(c) = a_{-\lambda_1}(c) a_{-\lambda_2}(c) \cdots$ for $c \in \Gamma_*$. For $\lambda = (\lambda(\gamma))_{\gamma \in \Gamma^*} \in \mathcal{OP}(\Gamma^*)$, we define

$$a_{-\lambda} = \prod_{\gamma \in \Gamma^*} a_{-\lambda(\gamma)}(\gamma).$$

and similarly $a'_{-\rho} = \prod_{c \in \Gamma_*} a_{-\rho(c)}(c)$ for $\rho = (\rho(c))_{c \in \Gamma_*} \in \mathcal{OP}(\Gamma_*)$. It is clear that both $\{a_{-\lambda}\}$ and $\{a'_{-\rho}\}$ are \mathbb{C} -bases for $S_{H\Gamma}^-$.

It follows from Eqn. (3.3) and (3.4) that

$$(3.5) \quad \langle a'_{-\rho'}, a'_{-\bar{\rho}} \rangle'_{\xi} = \delta_{\rho', \rho} \frac{Z_{\rho}}{2^{l(\rho)}} \prod_{c \in \Gamma_*} \xi(c)^{l(\rho(c))}, \quad \rho', \rho \in \mathcal{OP}(\Gamma_*).$$

The bilinear form $\langle \cdot, \cdot \rangle'_{\xi}$ induces a bilinear form on $\overline{S}_{H\Gamma}$ which will be denoted by the same notation.

3.4. The characteristic map ch . We define the *characteristic* map $ch : R_{H\Gamma}^- \longrightarrow S_{H\Gamma}^-$ by letting (compare with [FJW2])

$$(3.6) \quad ch(f) = \sum_{\rho \in \mathcal{OP}(\Gamma_*)} \frac{1}{Z_\rho} f_\rho a'_{-\bar{\rho}},$$

where $f_\rho = f(D_\rho^+)$. In the case when Γ is trivial, this is essentially the same as defined in [Jo2], and a different approach is given in [W2].

For $c \in \Gamma^*$ and $n \in 2\mathbb{Z} + 1$, we let $c_n(c \in \Gamma_*)$ be the split conjugacy class D_ρ^+ in $\tilde{H}\Gamma_n$ such that $\rho(c) = (n)$ and $\rho(c') = \emptyset$ for $c' \neq c$. Denote by $\sigma_n(c)$ the super class function on $\tilde{H}\Gamma_n$ which takes value $n\zeta_c$ on elements in the conjugacy class c_n and 0 elsewhere. For $\rho = \{(r^{m_r(c)})\}_{r \geq 1, c \in \Gamma_*} \in \mathcal{OP}_n(\Gamma_*)$, $\sigma_\rho = \prod_{r \geq 1, c \in \Gamma_*} \sigma_r(c)^{m_r(c)}$ is the class function on split conjugacy classes in $\tilde{H}\Gamma_n$ which takes value Z_ρ on the conjugacy class D_ρ^+ and 0 elsewhere. Given $\gamma \in R(\Gamma)$, we denote by $\sigma_n(\gamma)$ the class function on $\tilde{H}\Gamma_n$ which takes value $n\gamma(c)$ on elements in the class $c_n, c \in \Gamma_*$, and 0 elsewhere.

The following lemma follows from definitions.

Lemma 3.1. *The map ch sends σ_ρ to $a'_{-\rho}$. In particular, it sends $\sigma_n(c)$ to $a_{-n}(c)$ in $S_{H\Gamma}^-$ while sending $\sigma_n(\gamma)$ to $a_{-n}(\gamma)$.*

3.5. Action of $\hat{\mathfrak{h}}_{\Gamma, \xi}[-1]$ on $R_{H\Gamma}^-$. We define $\tilde{a}_{-n}(\gamma), n \in 2\mathbb{Z}_+ + 1$ to be a map from $R_{H\Gamma}^-$ to itself by the following composition

$$R^-(\tilde{H}\Gamma_m) \xrightarrow{\sigma_n(\gamma)^\otimes} R^-(\tilde{H}\Gamma_n) \bigotimes R^-(\tilde{H}\Gamma_m) \xrightarrow{Ind} R^-(\tilde{H}\Gamma_{n+m}).$$

We also define $\tilde{a}_n(\gamma), n \in 2\mathbb{Z}_+ + 1$ to be a map from $R_{H\Gamma}^-$ to itself as the composition

$$R^-(\tilde{H}\Gamma_m) \xrightarrow{Res} R^-(\tilde{H}\Gamma_n) \bigotimes R^-(\tilde{H}\Gamma_{m-n}) \xrightarrow{\langle \sigma_n(\gamma), \cdot \rangle_\xi} R^-(\tilde{H}\Gamma_{m-n}).$$

We denote by $R_{H\Gamma}^0$ the radical of the bilinear form $\langle \cdot, \cdot \rangle_\xi$ in $R_{H\Gamma}^-$ and denote by $\overline{R}_{H\Gamma}$ the quotient $R_{H\Gamma}^-/R_{H\Gamma}^0$, which inherits the bilinear form $\langle \cdot, \cdot \rangle_\xi$ from $R_{H\Gamma}^-$. The following theorem is implied by the identification of the Hopf algebra structures on $R_{H\Gamma}^-$ and $S_{H\Gamma}^-$ (see Proposition 3.2 below).

Theorem 3.2. *$R_{H\Gamma}^-$ is a module over the twisted Heisenberg algebra $\hat{\mathfrak{h}}_{\Gamma, \xi}[-1]$ by letting $a_n(\gamma)$ ($n \in 2\mathbb{Z} + 1$) act as $\tilde{a}_n(\gamma)$ and C as 1. $R_{H\Gamma}^0$ is a submodule of $R_{H\Gamma}^-$ over $\hat{\mathfrak{h}}_{\Gamma, \xi}[-1]$ and the quotient $\overline{R}_{H\Gamma}$ is irreducible. The characteristic map ch is an isomorphism of $R_{H\Gamma}^-$ (resp. $R_{H\Gamma}^0, \overline{R}_{H\Gamma}$) and $S_{H\Gamma}^-$ (resp. $S_{H\Gamma}^0, \overline{S}_{H\Gamma}$) as supermodules over $\hat{\mathfrak{h}}_{\Gamma, \xi}[-1]$.*

3.6. The character $\chi_n(\gamma)$. Recall that we have defined a map from $R(\Gamma)$ to $R^-(\tilde{H}\Gamma_n)$ by sending γ to $\chi_n(\gamma) := \eta_n(\gamma) \otimes \chi_n$, where χ_n is the character of the basic spin supermodule L_n . The image of $\chi_n(\gamma)$ has the following elegant description under the characteristic map in terms of a generating function in a formal variable z .

Proposition 3.1. *For any $\gamma \in R(\Gamma)$, we have*

$$(3.7) \quad \sum_{n \geq 0} \text{ch}(\chi_n(\gamma)) z^n = \exp \left(\sum_{n \geq 1, \text{odd}} \frac{2}{n} a_{-n}(\gamma) z^n \right).$$

Proof. Let γ be a character of Γ . It follows from Corollary 2.2 that

$$\begin{aligned} \sum_{n \geq 0} \text{ch}(\chi_n(\gamma)) z^n &= \sum_{\rho \in \mathcal{O}P_n(\Gamma_*)} 2^{l(\rho)} Z_\rho^{-1} \prod_{c \in \Gamma_*} \gamma(c)^{l(\rho(c))} a'_{-\rho(c)} z^{||\rho||} \\ &= \prod_{c \in \Gamma_*} \left(\sum_{\lambda \in \mathcal{O}P_n} (2\zeta_c^{-1} \gamma(c))^{l(\lambda)} z_\lambda^{-1} a_{-\lambda}(c) z^{|\lambda|} \right) \\ &= \exp \left(\sum_{n \geq 1, \text{odd}} \frac{2}{n} \sum_{c \in \Gamma_*} \zeta_c^{-1} \gamma(c) a_{-n}(c) z^n \right) \\ &= \exp \left(\sum_{n \geq 1, \text{odd}} \frac{2}{n} a_{-n}(\gamma) z^n \right). \end{aligned}$$

It follows from Eq. (2.5) that $\sum_{n \geq 0} \text{ch}(\chi_n(\gamma)) z^n$ is multiplicative on γ .

Thus given two characters β, γ of Γ , we have

$$\begin{aligned} \sum_{n \geq 0} \text{ch}(\chi_n(\beta - \gamma)) z^n &= \sum_{n \geq 0} \text{ch}(\chi_n(\beta)) z^n \sum_{n \geq 0} \text{ch}(\chi_n(-\gamma)) z^n \\ &= \exp \left(\sum_{n \geq 1, \text{odd}} \frac{2}{n} a_{-n}(\beta) z^n \right) \exp \left(- \sum_{n \geq 1, \text{odd}} \frac{2}{n} a_{-n}(\gamma) z^n \right) \\ &= \exp \left(\sum_{n \geq 1, \text{odd}} \frac{2}{n} a_{-n}(\beta - \gamma) z^n \right). \end{aligned}$$

Therefore the proposition is proved. \square

Corollary 3.3. *The formula (2.6) holds for any $\gamma \in R(\Gamma)$. In particular $\chi_n(\xi)$ is self-dual if ξ is self-dual.*

Component-wise, we obtain

$$\text{ch}(\chi_n(\gamma)) = \sum_{\rho} \frac{2^{l(\rho)}}{z_{\rho}} a_{-\rho}(\gamma),$$

where the sum is over all the partitions ρ of n into odd integers.

3.7. Isometry between $R_{H\Gamma}^-$ and $S_{H\Gamma}^-$. In Sect. 2.4 we introduced the Hopf algebra structure on $R_{H\Gamma}^-$. On the other hand it is well known that there exists a natural Hopf algebra structure on the symmetric algebra $S_{H\Gamma}^-$ with the usual multiplication and the comultiplication Δ given by

$$(3.8) \quad \Delta(a_n(\gamma)) = a_n(\gamma) \otimes 1 + 1 \otimes a_n(\gamma), \quad n \in 2\mathbb{Z} + 1.$$

Proposition 3.2. *The characteristic map $ch : R_{H\Gamma}^- \longrightarrow S_{H\Gamma}^-$ is an isomorphism of Hopf algebras.*

Proof. First the map ch is a vector space isomorphism by comparing dimension. The algebra isomorphism follows from the Frobenius reciprocity. On the other hand, one can check directly that

$$\Delta(\sigma_n(\gamma)) = \sigma_n(\gamma) \otimes 1 + 1 \otimes \sigma_n(\gamma), \quad \gamma \in \Gamma^*, n \in 2\mathbb{Z} + 1.$$

Since $a_n(\gamma)$ (resp. $\sigma_n(\gamma)$) for $\gamma \in \Gamma^*, n \in 2\mathbb{Z} + 1$ generate $S_{H\Gamma}^-$ (resp. $R_{H\Gamma}^-$) as an algebra, we conclude that ch is a Hopf algebra isomorphism by (3.8). \square

Recall that we have defined a bilinear form $\langle \cdot, \cdot \rangle_{\xi}$ on $R_{H\Gamma}^-$ and a bilinear form $\langle \cdot, \cdot \rangle'_{\xi}$ on $S_{H\Gamma}^-$, and thus an induced one on $S_{H\Gamma}^- \otimes S_{H\Gamma}^-$. The lemma below follows from our definition of $\langle \cdot, \cdot \rangle'_{\xi}$ and the comultiplication Δ .

Lemma 3.4. *The bilinear form $\langle \cdot, \cdot \rangle'_{\xi}$ on $S_{H\Gamma}^-$ can be characterized by the following two properties:*

- 1). $\langle a_{-n}(\beta), a_{-m}(\gamma) \rangle'_{\xi} = \frac{n}{2} \delta_{n,m} \langle \beta, \gamma \rangle'_{\xi}, \quad \beta, \gamma \in \Gamma^*, m, n \in 2\mathbb{Z} + 1.$
- 2). $\langle fg, h \rangle'_{\xi} = \langle f \otimes g, \Delta h \rangle'_{\xi},$ where $f, g, h \in S_{H\Gamma}^-.$

Theorem 3.5. *The characteristic map ch is an isometry from the space $(R_{H\Gamma}^-, \langle \cdot, \cdot \rangle_{\xi})$ to $(S_{H\Gamma}^-, \langle \cdot, \cdot \rangle'_{\xi})$.*

Proof. Let f and g be any two super class functions in $R^-(\tilde{H}\Gamma_n)$. By definition of the characteristic map (3.6) it follows that

$$\begin{aligned} \langle ch(f), ch(g) \rangle'_\xi &= \sum_{\rho, \rho' \in \mathcal{OP}_n(\Gamma_*)} \frac{1}{Z_\rho Z_{\rho'}} f(\rho) g(\rho') \langle a'_{-\rho}, a'_{-\rho'} \rangle'_\xi \\ &= \sum_{\rho, \rho' \in \mathcal{OP}_n(\Gamma_*)} \frac{1}{Z_\rho Z_{\rho'}} f(\rho) g(\rho') \frac{Z_\rho}{2^{l(\rho')}} \prod_{c \in \Gamma_*} \xi(c)^{l(\rho(c))} \delta_{\rho, \rho'} \\ &= \langle f, g \rangle_\xi, \end{aligned}$$

where we have used the inner product identity (3.5) and (2.14). \square

From now on the bilinear form $\langle \cdot, \cdot \rangle_\xi$ on $R_{H\Gamma}^-$ is identified with the one $\langle \cdot, \cdot \rangle'_\xi$ on $S_{H\Gamma}^-$.

4. VERTEX REPRESENTATIONS AND $R_{H\Gamma}^-$

In this section, we construct twisted vertex operators on a space $\mathcal{F}_{H\Gamma}^T$ obtained by $R_{H\Gamma}^-$ tensored with a certain group algebra constructed from the finite group Γ . In a most important case when Γ is a finite subgroup of $SL_2(\mathbb{C})$, we obtain by vertex operator techniques a twisted affine Lie algebra and toroidal Lie algebra acting on $\mathcal{F}_{H\Gamma}^T$ in terms of group-theoretic operators.

4.1. A central extension of $R_{\mathbb{Z}}(\Gamma)/2R_{\mathbb{Z}}(\Gamma)$. From now on we assume that ξ is a self-dual virtual character of Γ , then $R_{\mathbb{Z}}(\Gamma)$ is an integral lattice with the symmetric bilinear form $\langle \cdot, \cdot \rangle_\xi$.

The quotient lattice $R_{\mathbb{F}_2}(\Gamma) = R_{\mathbb{Z}}(\Gamma)/2R_{\mathbb{Z}}(\Gamma)$ is an $(r+1)$ -dimensional vector space over the field $\mathbb{F}_2 = \mathbb{Z}_2$. Write $\bar{\alpha} = \alpha + 2R_{\mathbb{Z}}(\Gamma)$. Then

$$c_1(\bar{\alpha}, \bar{\beta}) = \langle \alpha, \beta \rangle_\xi + \langle \alpha, \alpha \rangle_\xi \langle \beta, \beta \rangle_\xi \pmod{2}$$

is a natural (even) alternating form on $R_{\mathbb{F}_2}(\Gamma)$ and let r_0 be its rank over \mathbb{F}_2 . The alternating form c_1 defines a central extension $\hat{R}_{\mathbb{F}_2}^-(\Gamma)$ of the abelian group $R_{\mathbb{F}_2}(\Gamma)$ by the two-element group $\langle \pm 1 \rangle$ (cf. [FLM1]):

$$(4.1) \quad 1 \rightarrow \langle \pm 1 \rangle \hookrightarrow \hat{R}_{\mathbb{F}_2}^-(\Gamma) \xrightarrow{\sim} R_{\mathbb{F}_2}(\Gamma) \rightarrow 1,$$

such that $aba^{-1}b^{-1} = (-1)^{c_1(\check{a}, \check{b})}$, $a, b \in \hat{R}_{\mathbb{F}_2}^-(\Gamma)$. It is easily seen that $\{\pm e_{\bar{\alpha}} | \alpha \in R_{\mathbb{Z}}(\Gamma)\}$ form a basis for $\hat{R}_{\mathbb{F}_2}^-(\Gamma)$. We note that $(e_{\bar{\alpha}})^2 = 1$ and $\dim(\hat{R}_{\mathbb{F}_2}^-(\Gamma)) = 2^{r+2}$.

Let Φ be a subgroup of $R_{\mathbb{Z}}(\Gamma)$ which is maximal such that the alternating form c_1 vanishes on $\Phi/2R_{\mathbb{Z}}(\Gamma)$.

Lemma 4.1. [FLM2, FJW2] *There are $2^{(r+1-r_0)}$ irreducible $\hat{R}_{\mathbb{F}_2}^-(\Gamma)$ -module structures on the space $\mathbb{C}[R_{\mathbb{Z}}(\Gamma)/\Phi]$ such that $-1 \in \hat{R}_{\mathbb{F}_2}^-(\Gamma)$ acts faithfully and*

$$(4.2) \quad e_{\bar{\alpha}}e_{\bar{\beta}} = e_{\bar{\beta}}e_{\bar{\alpha}}(-1)^{c_1(\bar{\alpha}, \bar{\beta})}$$

as operators on $\mathbb{C}[R_{\mathbb{Z}}(\Gamma)/\Phi]$. Moreover, $\dim(\mathbb{C}[R_{\mathbb{Z}}(\Gamma)/\Phi]) = 2^{\frac{1}{2}r_0}$.

We will denote the elements of $\mathbb{C}[R_{\mathbb{Z}}(\Gamma)/\Phi]$ by $e^{[\alpha]}$, where $[\alpha] = \alpha + \Phi \in R_{\mathbb{Z}}(\Gamma)/\Phi$. Clearly

$$e^{2[\alpha]} = 1, \quad e^{[\alpha+\beta]} = e^{[\alpha]}e^{[\beta]}.$$

For $\alpha, \beta \in R_{\mathbb{Z}}(\Gamma)$ we write the action of $\hat{R}_{\mathbb{F}_2}^-(\Gamma)$ on $\mathbb{C}[R_{\mathbb{Z}}(\Gamma)/\Phi]$ as

$$(4.3) \quad e_{\bar{\alpha}} \cdot e^{[\beta]} = \epsilon(\alpha, \beta)e^{[\alpha+\beta]}.$$

Then one can check that ϵ is a well-defined cocycle map from $R_{\mathbb{Z}}(\Gamma) \times R_{\mathbb{Z}}(\Gamma) \rightarrow \langle \pm 1 \rangle$. One also has $\epsilon(\alpha, \beta) = \epsilon(\alpha, -\beta)$.

4.2. Twisted Vertex Operators $X(\gamma, z)$. We fix an irreducible $\hat{R}_{\mathbb{F}_2}^-(\Gamma)$ -module structure on $\mathbb{C}[R_{\mathbb{Z}}(\Gamma)/\Phi]$ described in Eqn. (4.3). We extend the actions of $e_{\bar{\alpha}}$ to the space of tensor product

$$\mathcal{F}_{H\Gamma}^T = R_{H\Gamma}^- \bigotimes \mathbb{C}[R_{\mathbb{Z}}(\Gamma)/\Phi],$$

by letting them act on the $R_{H\Gamma}^-$ part trivially.

Introduce the operators $H_{\pm n}(\gamma), \gamma \in R(\Gamma), n > 0$, as the following compositions of maps:

$$\begin{aligned} H_{-n}(\gamma) &: R^-(\tilde{H}\Gamma_m) \xrightarrow{\chi_n(\gamma) \otimes} R^-(\tilde{H}\Gamma_n \hat{\times} \tilde{H}\Gamma_m) \xrightarrow{Ind} R^-(\tilde{H}\Gamma_{n+m}) \\ H_n(\gamma) &: R^-(\tilde{H}\Gamma_m) \xrightarrow{Res} R^-(\tilde{H}\Gamma_n \hat{\times} \tilde{H}\Gamma_{m-n}) \xrightarrow{\langle \chi_n(\gamma), \cdot \rangle_{\xi}} R^-(\tilde{H}\Gamma_{m-n}). \end{aligned}$$

We define

$$H_+(\gamma, z) = \sum_{n>0} H_{-n}(\gamma) z^n, \quad H_-(\gamma, z) = \sum_{n>0} H_n(\gamma) z^{-n},$$

where z is a formal variable. We define the twisted vertex operators $X_n(\gamma), n \in \mathbb{Z}, \gamma \in R_{H\Gamma}$ by the following generating functions:

$$(4.4) \quad X(\gamma, z) = H_+(\gamma, z)H_-(\gamma, -z)e_{\bar{\gamma}} = \sum_{n \in \mathbb{Z}} X_n(\gamma) z^{-n}.$$

We note that $X(-\gamma, z) = X(\gamma, -z)$. The operators $X_n(\gamma)$ are well-defined operators acting on the space $\mathcal{F}_{H\Gamma}^T$. We extend the bilinear form $\langle \cdot, \cdot \rangle_\xi$ on $R_{H\Gamma}^-$ to $\mathcal{F}_{H\Gamma}^T$ by letting

$$\langle f e^{[\alpha]}, g e^{[\beta]} \rangle_\xi = \langle f, g \rangle_\xi \delta_{[\alpha], [\beta]}, \quad f, g \in R_{H\Gamma}^-, \alpha, \beta \in R_{\mathbb{Z}}(\Gamma).$$

We extend the \mathbb{Z}_+ -gradation \deg from $R_{H\Gamma}^-$ to $\mathcal{F}_{H\Gamma}^T$ by letting

$$\deg a_{-n}(\gamma) = n, \quad \deg e_{\overline{\gamma}} = 0.$$

Similarly we extend the bilinear form $\langle \cdot, \cdot \rangle_\xi$ to the space

$$V_{H\Gamma}^T = S_{H\Gamma}^- \otimes \mathbb{C}[R_{\mathbb{Z}}(\Gamma)/\Phi]$$

and extend the \mathbb{Z}_+ -gradation on $S_{H\Gamma}^-$ to a \mathbb{Z}_+ -gradation on $V_{H\Gamma}^T$.

The characteristic map ch will be extended to an isometry from $\mathcal{F}_{H\Gamma}^T$ to $V_{H\Gamma}^T$ by fixing the subspace $\mathbb{C}[R_{\mathbb{Z}}(\Gamma)/\Phi]$. We will denote this map again by ch .

4.3. An identity for twisted vertex operators. We extend the characteristic map ch to a linear map $ch: \text{End}(R_{H\Gamma}^-) \rightarrow \text{End}(S_{H\Gamma}^-)$ by

$$(4.5) \quad ch(f).ch(v) = ch(f.v), \quad f \in \text{End}(R_{H\Gamma}^-), v \in R_{H\Gamma}^-.$$

The relation between the vertex operators defined in (4.4) and the Heisenberg algebra $\widehat{\mathfrak{h}}_{\Gamma, \xi}$ is revealed in the following theorem.

Theorem 4.2. *For any $\gamma \in R(\Gamma)$, we have*

$$\begin{aligned} ch(H_+(\gamma, z)) &= \exp\left(\sum_{n \geq 1, \text{odd}} \frac{2}{n} a_{-n}(\gamma) z^n\right), \\ ch(H_-(\gamma, z)) &= \exp\left(\sum_{n \geq 1, \text{odd}} \frac{2}{n} a_n(\gamma) z^{-n}\right). \end{aligned}$$

Proof. The first identity follows from Proposition 3.1 and the second one is obtained by noting that $H_+(\gamma, z)$ is the adjoint operator of $H_-(\gamma, z^{-1})$ with respect to the bilinear form $\langle \cdot, \cdot \rangle_\xi$. \square

As a consequence we have

$$\begin{aligned} &ch(X(\gamma, z)) \\ &= \exp\left(\sum_{n \geq 1, \text{odd}} \frac{2}{n} a_{-n}(\gamma) z^n\right) \exp\left(-\sum_{n \geq 1, \text{odd}} \frac{2}{n} a_n(\gamma) z^{-n}\right) e_{\overline{\gamma}}. \end{aligned}$$

Thus the characteristic map identifies the twisted vertex operators $X(\gamma, z)$ defined via finite groups $\widetilde{H}\Gamma_n$ with the usual twisted vertex operators of [FLM1, FLM2].

4.4. Product of two vertex operators. The normal ordered product $:X(\alpha, z)X(\beta, w):$, $\alpha, \beta \in R(\Gamma)$ of two vertex operators is defined as follows:

$$:X(\alpha, z)X(\beta, w): = H_+(\alpha, z)H_+(\beta, w)H_-(\alpha, -z)H_-(\beta, -w)e_{\overline{\alpha+\beta}}.$$

The following theorem can be verified using the standard vertex operator calculus (see e.g. [FLM2, J1, FJW2]), where the term $\left(\frac{z-w}{z+w}\right)^{\langle\alpha, \beta\rangle_\xi}$ is understood as the power series expansion in the variable w/z .

Theorem 4.3. *For $\alpha, \beta \in R(\Gamma)$ one has the following operator product expansion identity for twisted vertex operators.*

$$X(\alpha, z)X(\beta, w) = \epsilon(\alpha, \beta) :X(\alpha, z)X(\beta, w): \left(\frac{z-w}{z+w}\right)^{\langle\alpha, \beta\rangle_\xi}.$$

The next proposition follows readily from Theorem 4.2.

Proposition 4.1. *Given $\alpha \in R(\Gamma)$, $\beta \in R_{\mathbb{Z}}(\Gamma)$ and $n \in 2\mathbb{Z} + 1$, we have*

$$[a_n(\alpha), X(\beta, z)] = \langle\alpha, \beta\rangle_\xi X(\beta, z)z^n.$$

4.5. Twisted affine algebra $\widehat{\mathfrak{g}}[-1]$ and toroidal algebra $\widehat{\widehat{\mathfrak{g}}}[-1]$. Let \mathfrak{g} be a rank r complex simple Lie algebra of ADE type, and let $\overline{\Delta}$ be the root system generated by a set of simple roots $\alpha_1, \dots, \alpha_r$. Let α_{max} be the highest root. The Lie algebra is generated by the Chevalley generators $e_{\alpha_i}, e_{-\alpha_i}, h_i = h_{\alpha_i}$. We normalize the invariant bilinear form on \mathfrak{g} by $(\alpha_{max}, \alpha_{max}) = 2$.

The twisted toroidal algebra $\widehat{\widehat{\mathfrak{g}}}[-1]$ (associated to \mathfrak{g}) is the associative algebra generated by ([J2, FJW2])

$$C, h_i(m), x_n(\pm\alpha_i), m \in 2\mathbb{Z} + 1, n \in \mathbb{Z}, i = 0, \dots, r;$$

subject to the relations: C is central, $x_n(\alpha_i) = (-1)^n x_n(-\alpha_i)$ and

$$\begin{aligned} [h_i(m), h_j(m')] &= \frac{m}{2} a_{ij} \delta_{m, -m'} C, \\ [h_i(n), x_m(\alpha_j)] &= a_{ij} x_{n+m}(\alpha_j), \\ (4.6) \quad [x_n(\alpha_i), x_{n'}(-\alpha_i)] &= 8\{h_i(n + n') + n\delta_{n, -n'} C\}, \\ \sum_{s=0}^{a_{ij}} \binom{a_{ij}}{s} [x_{n+s}(\alpha_i), x_{n'-a_{ij}-s}(\alpha_j)] &= 0, \quad \text{if } a_{ij} \geq 0 \\ \sum_{s=0}^{-a_{ij}} (-1)^s \binom{-a_{ij}}{s} [x_{n+s}(\alpha_i), x_{n'-a_{ij}-s}(\alpha_j)] &= 0, \quad \text{if } a_{ij} < 0 \end{aligned}$$

where $n, n' \in \mathbb{Z}$, $m, m' \in 2\mathbb{Z} + 1$, $i, j = 0, 1, \dots, r$, and $h_i(2n) = 0$ for $n \in \mathbb{Z}$.

The twisted affine algebra (cf. [FLM1, FJW2]), denoted by $\widehat{\mathfrak{g}}[-1]$, can be identified with the subalgebra of $\widehat{\mathfrak{g}}[-1]$ generated by

$$C, h_i(m), x_n(\pm\alpha_i), m \in 2\mathbb{Z} + 1, n \in \mathbb{Z}, i = 1, \dots, r;$$

The *basic twisted representation* V of $\widehat{\mathfrak{g}}[-1]$ is the irreducible highest weight representation generated by a highest weight vector which is annihilated by $h_i(m)(m \in 2\mathbb{Z}_+ + 1), x_n(\pm\alpha_i)(n \in \mathbb{Z}_+)$, and C acts on V as the identity operator.

4.6. Realization of the twisted representations. Let Γ be a finite subgroup of $SL_2(\mathbb{C})$ and the virtual character ξ to be twice the trivial character minus the character of the two-dimensional defining representation of $\Gamma \hookrightarrow SL_2(\mathbb{C})$. The following is the well-known list of finite groups of $SL_2(\mathbb{C})$: the cyclic, binary dihedral, tetrahedral, octahedral and icosahedral groups. McKay observed that the associated matrix to the weighted bilinear form $\langle -, - \rangle_\xi$ on $R(\Gamma)$ with respect to the basis of irreducible characters can be identified with an affine Dynkin diagram of ADE type [Mc].

The following theorem provides a finite group realization of the vertex representation of the twisted toroidal Lie algebra $\widehat{\mathfrak{g}}[-1]$ on $\mathcal{F}_{H\Gamma}^T$.

Theorem 4.4. *A vertex representation of the twisted toroidal Lie algebra $\widehat{\mathfrak{g}}[-1]$ is defined on the space $\mathcal{F}_{H\Gamma}^T$ by letting*

$$\begin{aligned} x_n(\alpha_i) &\mapsto X_n(\gamma_i), & x_n(-\alpha_i) &\mapsto \epsilon(\gamma_i, \gamma_i) X_n(-\gamma_i), \\ h_i(m) &\mapsto a_m(\gamma_i), & C &\mapsto 1, \end{aligned}$$

where $n \in \mathbb{Z}, m \in 2\mathbb{Z} + 1, 0 \leq i \leq r$.

Proof. All the commutation relations without binomial coefficients are easy consequences of Proposition 4.1 and Theorem 4.3 by the usual vertex operator calculus in the twisted picture (see [FLM2, J1]). The corresponding relations with binomial coefficients in $V_{H\Gamma}^T$ are equivalent to

$$\begin{aligned} (z + w)^{a_{ij}} [X(\gamma_i, z), X(\gamma_j, w)] &= 0, & a_{ij} &\geq 0, \\ (z - w)^{-a_{ij}} [X(\gamma_i, z), X(\gamma_j, w)] &= 0, & a_{ij} &> 0. \end{aligned}$$

This is proved by using Theorem 4.3 by a standard method in the theory of vertex algebras (cf. e.g. [FLM2, J2]). \square

Note that the quotient lattice $R_{\mathbb{Z}}(\Gamma)/R_{\mathbb{Z}}^0$ inherits a positive definite integral bilinear form from that of $R_{\mathbb{Z}}(\Gamma)$. Let $\overline{\Gamma}^* = \{\gamma_1, \gamma_2, \dots, \gamma_r\}$

be the set of non-trivial irreducible characters of Γ . Let $R_{\mathbb{Z}}(\bar{\Gamma}^*)$ be the sublattice of $R_{\mathbb{Z}}(\Gamma)$ generated by $\bar{\Gamma}^*$. Denote by $Sym(\bar{\Gamma}^*)$ the symmetric algebra generated by $a_{-n}(\gamma_i)$, $n \in 2\mathbb{Z}_+ + 1$, $i = 1, \dots, r$, which is isometric to $\bar{S}_{H\Gamma}$ and $\bar{R}_{H\Gamma}$ as well. The irreducible $\hat{R}_{\mathbb{F}_2}^-(\Gamma)$ -module $\mathbb{C}[R_{\mathbb{Z}}(\Gamma)/\Phi]$ induces an irreducible $\hat{R}_{\mathbb{F}_2}^-(\bar{\Gamma}^*)$ -module structure on $\mathbb{C}[R_{\mathbb{Z}}(\bar{\Gamma}^*)/\bar{\Phi}]$ given by the restriction of c_1 . Denote by \bar{r}_0 its rank. In this case if the determinant of the Cartan matrix is an odd integer (see Lemma 4.1), then $\bar{r}_0 = 0$ and the space $\mathbb{C}[R_{\mathbb{Z}}(\bar{\Gamma}^*)/\bar{\Phi}]$ is trivial.

We define

$$\begin{aligned}\bar{V}_{H\Gamma}^T &= \bar{S}_{H\Gamma} \bigotimes \mathbb{C}[R_{\mathbb{Z}}(\bar{\Gamma}^*)/\bar{\Phi}] \cong Sym(\bar{\Gamma}^*) \bigotimes \mathbb{C}[R_{\mathbb{Z}}(\bar{\Gamma}^*)/\bar{\Phi}], \\ \bar{\mathcal{F}}_{H\Gamma}^T &= \bar{R}_{H\Gamma}^T \bigotimes \mathbb{C}[R_{\mathbb{Z}}(\bar{\Gamma}^*)/\bar{\Phi}].\end{aligned}$$

Obviously ch restricted to $\bar{\mathcal{F}}_{H\Gamma}^T$ is an isometric isomorphism onto $\bar{V}_{H\Gamma}^T$. We remark that $\bar{\mathcal{F}}_{H\Gamma}^T$ is isomorphic to the tensor product of the space $\bar{\mathcal{F}}_{H\Gamma}$ associated to $\bar{R}_{\mathbb{Z}}(\Gamma)$ and the space associated to the rank 1 lattice $\mathbb{Z}\delta$ equipped with the zero bilinear form.

The identity for a product of vertex operators $X(\gamma, z)$ associated to $\gamma \in \bar{\Delta}$ (cf. Theorem 4.4) implies that $\bar{V}_{H\Gamma}^T$ provides a realization of the vertex representation of $\hat{\mathfrak{g}}[-1]$ on $\bar{V}_{H\Gamma}^T$ (cf. [FLM1]). The following theorem establishes a direct link from the finite group $\Gamma \in SL_2(\mathbb{C})$ to the affine Lie algebra $\hat{\mathfrak{g}}[-1]$.

Theorem 4.5. *The operators $X_n(\gamma)$, $\gamma \in \bar{\Delta}$, $a_n(\gamma_i)$, $i = 1, 2, \dots, r$, $n \in \mathbb{Z}$ define an irreducible representation of the affine Lie algebra $\hat{\mathfrak{g}}[-1]$ on $\bar{\mathcal{F}}_{H\Gamma}^T$ which is isomorphic to the twisted basic representation.*

5. THE CHARACTER TABLES OF $\tilde{H}\Gamma_n$ AND VERTEX OPERATORS

In this section we outline how to use the specialization of $\xi = \gamma_0$ to obtain the character table for the spin supermodules of $\tilde{H}\Gamma_n$ from our vertex operator approach. The proofs are similar to those given in [FJW2] (also cf. [J1]) which we omit.

5.1. Components of vertex operators. Set $\xi = \gamma_0$. The weighted bilinear form reduces to the standard one and $R_{\mathbb{Z}}(\Gamma) \simeq \mathbb{Z}^{r+1}$.

Let $A = (1 - \delta_{ij})_{i,j=0}^r$, the matrix of the alternating form c_1 over \mathbb{F}_2 , then $A^2 = \bar{r}I$. Here $\bar{r} = 0$ if r is even and 1 if r is odd. Consequently Lemma 4.1 implies there are exactly 2^{r+1} irreducible $\hat{R}_{\mathbb{F}_2}^-(\Gamma)$ -module structures on the $2^{\lceil \frac{r+1}{2} \rceil}$ -dimensional space $\mathbb{C}[R_{\mathbb{Z}}(\Gamma)/\Phi]$, where $\lceil \frac{r+1}{2} \rceil$ denotes the smallest integer $\leq \frac{r+1}{2}$. One of the (at most) two irreducible

module structures is given by the cocycle $\epsilon(\gamma_i, \gamma_j) = 1$, for $i \leq j$, and $\epsilon(\gamma_i, \gamma_j) = -1$, for $i > j$.

In the following result the bracket $\{ , \}$ denotes the anti-commutator.

Theorem 5.1. *The operators $X_n^+(\gamma_i), X_n^-(\gamma_i)$ ($n \in \mathbb{Z}, 0 \leq i \leq r$) generate a Clifford algebra:*

$$(5.1) \quad \begin{aligned} \{X_n(\gamma_i), X_{n'}(\gamma_j)\} &= 2(-1)^n \delta_{ij} \delta_{n, -n'}, \\ \{X_n(-\gamma_i), X_{n'}(-\gamma_j)\} &= 2(-1)^n \delta_{ij} \delta_{n, -n'}, \\ \{X_n(\gamma_i), X_{n'}(-\gamma_j)\} &= 2\delta_{ij} \delta_{n, -n'}. \end{aligned}$$

Proof. The relations can be established using Theorem 4.3 by means of standard techniques in the theory of vertex algebras, see [J1] for detail for a similar circumstance. \square

5.2. Spin character tables of $\tilde{H}\Gamma_n$ and vertex operators. Let $R_{\mathbb{Z}}^-(\tilde{H}\Gamma_n)$ be the lattice generated by the characters of spin irreducible $\tilde{H}\Gamma_n$ -supermodules. Then $R_{\mathbb{Z}}^-(\tilde{H}\Gamma_n) \otimes \mathbb{C} \simeq R^-(\tilde{H}\Gamma_n)$.

For an m -tuple index $\phi = (\phi_1, \dots, \phi_m) \in \mathbb{Z}^m$ we denote

$$X_\phi(\gamma) = X_{\phi_1}(\gamma) \cdots X_{\phi_m}(\gamma) e^{[\alpha]}, \quad x_\phi(\gamma) = (X_{\phi_1}(\gamma).1) \cdots (X_{\phi_m}(\gamma).1).$$

The space $R_{H\Gamma}^-$ is spanned by $X_\phi(\gamma)$. However $X_{-n}(\pm\gamma).e^{[\alpha]} = 0$ if $n < 0$. Using the Clifford algebra structure we know that $R_{H\Gamma}^-$ is spanned by $X_{\phi, [\alpha]}(\gamma)$, where ϕ runs through m -tuples $\phi = (\phi_1, \dots, \phi_m) \in \mathbb{Z}^m$ and $[\alpha] \in R_{\mathbb{Z}}(\Gamma)/\Phi$. We define the raising operator R_{ij} by

$$R_{ij}(\phi_1, \dots, \phi_m) = (\phi_1, \dots, \phi_i + 1, \dots, \phi_j - 1, \dots, \phi_m).$$

Then we define the action of the raising operator R_{ij} on $X_{\phi, [\alpha]}(\gamma)$ or $x_\phi(\gamma)$ by $X_{R_{ij}\phi, [\alpha]}(\gamma)$ or $x_{R_{ij}\phi}(\gamma)$ respectively.

Given $\lambda \in \mathcal{OP}(\Gamma^*)$, we define

$$X_\lambda = \prod_{i=0}^r X_{-\lambda(\gamma_i)}(\gamma_i).$$

Similarly we write $x_\lambda = \prod_{\gamma \in \Gamma^*} x_{\lambda(\gamma)}(\gamma)$.

Proposition 5.1. [FJW2] *The vectors $X_\lambda e^{[\alpha]}$ for $\lambda = (\lambda(\gamma))_{\gamma \in \Gamma^*} \in \mathcal{SP}(\Gamma^*)$ and $[\alpha] \in R_{\mathbb{Z}}(\Gamma)/\Phi$ form an orthogonal basis in the vector space $R_{H\Gamma}^-$ with $\langle X_\lambda e^{[\alpha]}, X_\mu e^{[\beta]} \rangle = 2^{l(\lambda)} \delta_{\lambda, \mu} \delta_{[\alpha], [\beta]}$. Moreover, we have that*

$$(5.2) \quad X_\lambda e^{[\alpha]} = \prod_{\gamma \in \Gamma^*} \prod_{ij} \frac{1 - R_{ij}}{1 + R_{ij}} x_\lambda e^{l(\lambda)[\alpha]} = (x_\lambda + \sum_{\lambda \gg \mu} c_{\lambda, \mu} x_\mu) e^{l(\lambda)[\alpha]},$$

where $c_{\lambda,\mu} \in \mathbb{Z}$, and R_{ij} is the raising operator.

We remark that the basis elements in $S_{H\Gamma}^-$ corresponding to x_λ are the classical symmetric functions Q_λ called Schur's Q-functions [M, J1]. If we take $a_{-n}(\gamma)$, $n > 0$, $\gamma \in \Gamma^*$ as the n -th power sum in a sequence of variables $y_\gamma = (y_{i\gamma})_{i \geq 1}$, then the space $S_{H\Gamma}^-$ becomes a distinguished subspace of symmetric functions generated by odd degree power sums indexed by Γ^* . In particular $Q_\lambda(\gamma)$ is the Schur's Q-function in the variables y_γ . For $\lambda \in \mathcal{P}(\Gamma^*)$, we denote

$$(5.3) \quad Q_\lambda = \prod_{\gamma \in \Gamma^*} Q_{\lambda(\gamma)}(\gamma) \in S_{H\Gamma}^-.$$

For $\lambda \in \mathcal{SP}_n(\Gamma^*)$, we define $\tilde{H}\Gamma_\lambda = \tilde{H}\Gamma_{\lambda(\gamma_0)} \hat{\times} \cdots \hat{\times} \tilde{H}\Gamma_{\lambda(\gamma_r)}$. For a partition μ and an irreducible character γ of Γ , we define the spin character $\chi_\mu(\gamma)$ of $\tilde{H}\Gamma_\mu$ to be $\chi_{\mu_1}(\gamma) \otimes \cdots \otimes \chi_{\mu_l}(\gamma)$ (see Corollary 2.2).

Theorem 5.2. *For each strict partition-valued function $\lambda \in \mathcal{SP}_n(\Gamma^*)$, the vector $2^{-[\ell(\lambda)/2]} Q_\lambda$ corresponds, under the characteristic map ch , to the irreducible character χ_λ of the spin $\tilde{H}\Gamma_n$ -supermodule given by a \mathbb{Z} -linear combination of*

$$(5.4) \quad \text{Ind}_{\tilde{H}\Gamma_\rho}^{\tilde{H}\Gamma_n} \chi_{\rho(\gamma_0)}(\gamma_0) \otimes \cdots \otimes \chi_{\rho(\gamma_r)}(\gamma_r),$$

where $\rho \ll \lambda$ and the first summand is $\rho = \lambda$ with multiplicity one. Its character at the conjugacy class of type (μ, \emptyset) , where $\mu \in \mathcal{OP}_n(\Gamma_*)$, is equal to the matrix coefficient

$$(5.5) \quad 2^{\ell(\mu) - [\ell(\lambda)/2]} \langle X_\lambda e^{-[\lambda]}, a'_{-\mu} \rangle,$$

where $[\lambda] = \sum_{i=0}^r \ell(\lambda(\gamma_i))[\gamma_i]$. Moreover the degree of the character is equal to

$$(5.6) \quad 2^{n - [\ell(\lambda)/2]} n! \prod_{\gamma \in \Gamma^*} \left(\frac{\deg(\gamma)^{|\lambda(\gamma)|}}{\prod_{1 \leq i \leq \ell(\lambda(\gamma))} \lambda_i(\gamma)!} \prod_{ij} \frac{\lambda_i(\gamma) - \lambda_j(\gamma)}{\lambda_i(\gamma) + \lambda_j(\gamma)} \right).$$

All the irreducible spin $\tilde{H}\Gamma_n$ -supermodules can be described easily as follows. For each irreducible character $\gamma \in \Gamma^*$ let U_γ be the irreducible Γ -module affording γ . For each strict partition ν of n let V_ν be the corresponding irreducible spin supermodule of \tilde{H}_n (cf. [Sg, Jo2]). We have seen earlier that $U_\gamma^{\otimes n} \otimes V_\nu$ is an irreducible spin $\tilde{H}\Gamma_n$ -supermodule.

Proposition 5.2. *For each strict partition-valued function $\lambda = (\lambda(\gamma)) \in \mathcal{SP}_n(\Gamma^*)$, the tensor product*

$$\prod_{\gamma \in \Gamma^*} (U_{\gamma}^{\otimes l(\lambda(\gamma))} \otimes V_{\lambda(\gamma)})$$

decomposes completely into $2^{\lceil m/2 \rceil}$ copies of an irreducible spin $\tilde{H}\Gamma_{\lambda}$ -supermodule, where m denotes the number of the partitions of odd length among $\lambda(\gamma)$. Denote this irreducible module by W_{λ} . Then the induced supermodule $\text{Ind}_{\tilde{H}\Gamma_{\lambda}}^{\tilde{H}\Gamma_n} W_{\lambda}$ is the irreducible spin $\tilde{H}\Gamma_n$ -supermodule corresponding to λ , and it is of type M or Q according to $l(\lambda)$ is even or odd.

REFERENCES

- [FJW1] I. B. Frenkel, N. Jing and W. Wang, *Vertex representations via finite groups and the McKay correspondence*, Internat. Math. Res. Notices, No. **4** (2000) 195–222.
- [FJW2] I. B. Frenkel, N. Jing and W. Wang, *Twisted vertex representations via spin groups and the McKay correspondence*, math.QA/0007159, Duke J. Math., to appear.
- [FLM1] I. B. Frenkel, J. Lepowsky and A. Meurman, *An E_8 -approach to F_1* , Contemp. Math. **45** (1985) 99–120.
- [FLM2] I. B. Frenkel, J. Lepowsky and A. Meurman, *Vertex operator algebras and the Monster*, Academic Press, New York, 1988.
- [J1] N. Jing, *Vertex operators, symmetric functions and the spin group Γ_n* , J. Alg. **138** (1991) 340–398.
- [J2] N. Jing, *New twisted quantum current algebras*, Representations and quantizations (Shanghai, 1998), China High. Educ. Press, Beijing and Springer, New York, 2000, pp. 263–274. (math.QA/9901066)
- [Jo1] T. Józefiak, *Semisimple superalgebras*, In: Algebra—Some Current Trends (Varna, 1986), pp. 96–113, Lect. Notes in Math. **1352**, Springer-Berlag, Berlin-New York, 1988.
- [Jo2] T. Józefiak, *A class of projective representations of hyperoctahedral groups and Schur Q -functions*, Topics in Algebra, Banach Center Publ., **26**, Part 2, PWN-Polish Scientific Publishers, Warsaw(1990) 317–326.
- [LW] J. Lepowsky and R. L. Wilson, *Construction of the affine Lie algebra $A_1^{(1)}$* , Commun. Math. Phys. **62** (1978), 43–53.
- [M] I. G. Macdonald, *Symmetric functions and Hall polynomials*, 2nd ed., Clarendon Press, Oxford, 1995.
- [Mc] J. McKay, *Graphs, singularities and finite groups*, Proc. Sympos. Pure Math. **37** (1980) 183–186, Amer. Math. Soc, Providence, RI.
- [Naz] M. Nazarov, *Young’s symmetrizers for projective representations of the symmetric group*, Adv. in Math. **127** (1997) 190–257.
- [R] E. W. Read, *The α -regular classes of the generalized symmetric groups*, Glasgow Math. J. **17** (1976) 144–150.

- [S] I. Schur, *Über die Darstellung der symmetrischen und der alternierenden Gruppe durch gebrochene lineare Substitutionen*, J. Reine Angew. Math. **139** (1911) 155-250.
- [Sg] A. N. Sergeev, *The tensor algebra of the identity representation as a module over the Lie superalgebras $\mathfrak{gl}(m, n)$ and $Q(n)$* , Math. USSR Sbornik **51** (1985) 419-425.
- [St] J. R. Stembridge, *The projective representations of the hyperoctahedral group*, J. Alg. **145** (1992) 396-453.
- [W1] W. Wang, *Equivariant K-theory and wreath products*, MPI preprint # **86**, August 1998; *Equivariant K-theory, wreath products and Heisenberg algebra*, Duke Math. J. **103** (2000) 1-23. (math.QA/9907151)
- [W2] W. Wang, *Equivariant K-theory, generalized symmetric products, and twisted Heisenberg algebra*, Preprint, submitted.
- [Y] M. Yamaguchi, *A duality of the twisted group algebra of the symmetric group and a Lie superalgebra*, J. Alg. **222** (1999) 301-327.
- [Z] A. Zelevinsky, *Representations of finite classical groups, A Hopf algebra approach*. Lect. Notes in Math. **869**, Springer-Verlag, Berlin-New York, 1981.

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